A Tutorial on Gröbner Bases With Applications in Signals and Systems

Zhiping Lin, *Senior Member, IEEE*, Li Xu, *Member, IEEE*, and Nirmal K. Bose, *Life Fellow, IEEE*

(Invited Paper)

***Abstract*—This paper is a tutorial on Gröbner bases and a survey on the applications of Gröbner bases in the broad field of signals and systems. A reasonably detailed review is given of several fundamental theoretical issues that occur in the use of Gröbner bases in multidimensional signals and systems applications. These topics include the primeness of multivariate polynomial matrices, multivariate unimodular polynomial matrix completion, and prime factorization of multivariate polynomial matrices. A brief review is also presented on the wide-ranging applications of Gröbner bases in multidimensional as well as one-dimensional circuits, networks, control, coding, signals, and systems and other related areas like robotics and applied mechanics. The impact and scope of Gröbner bases in signals and systems are highlighted with respect to what has already been accomplished as a stepping stone to expanding future research.**

***Index Terms*—Circuits and systems, Gröbner bases, matrix factorization, multidimensional systems, multivariate polynomial, signal processing.**

I. Introduction

THE algorithmic algebra of Gröbner bases was developed by Buchberger in the 1960s and further enriched with contributions from him and many other researchers (see, e.g., [15]–[19], [55], and the references therein). Gröbner bases theory and its predecessors (like standard bases of Hironaka [50]) provide a rich theoretical framework in algebraic geometry and commutative algebra, particularly polynomial ideal theory. The algorithmic algebra of Gröbner bases has wide-ranging applications in theoretical physics, applied science, and engineering. The main reason for the success of Gröbner bases is that many problems in mathematics, science, and engineering can be represented by multivariate polynomials (e.g., ideals, modules, and matrices), where Gröbner bases play a role similar to the role of Euclidean Division Algorithm in the Euclidean ring of univariate polynomials. In 2001, a special issue on the applications of Gröbner bases to multidimensional systems and signal processing, published in *Multidimensional Systems and Signal Processing*, was guest-edited by two of the authors to emphasize the useful role of Gröbner bases to researchers in the area of multidimensional (n-D) systems theory and signal processing [65]. A special issue on multidimensional signals and systems, published in this journal [5], substantiated the increasing scope of the subject-matter. The present paper intends to respond to the accelerated need of Gröbner bases to these momentous developments and to attract new researchers with new applications that may prove to be a fertile ground for implementing the developments in algorithmic polynomial ideal theory.

Previously, symbolic algebra (coupled with numerical algebra), which is indispensable in the formulation and solving of many problems in circuits, systems, and signal processing, was implemented using software like REDUCE (Stanford University) and SAC (University of Wisconsin) [9]. Alternate software for the purpose like MACSYMA (M. I. T.), MAPLE (University of Waterloo), MATHEMATICA (Wolfram Research), and SCRATCHPAD (IBM) also existed. Algorithms developed to test for properties like stability of continuous and discrete dynamic systems and multivariate polynomial positivity were based either on the algebraic theory of resultants-subresultants (inners, bigradients) [9] and Bezoutiants [12]. With the development and gradual popularization of the algorithmic theory of Gröbner bases, not only did theoretical machineries require perusal of polynomial ideal and module theory but also implementation issues required the development of software like SINGULAR [116], COCOA [30], and MACAULAY [74]. Currently, these tools are finding wide-ranging applications of interest to readers of this journal with the promise of many more to come.

With the above background, the objective of this paper is threefold. First, we give a tutorial on Gröbner bases. Although many books and several tutorial papers on Gröbner bases are available in the literature, their targeted readers are mainly mathematicians or computer scientists rather than practicing electrical engineers in circuits, signals, and systems, who comprise the main targeted readers of this journal. In fact, although Gröbner bases have been widely applied in many areas of circuits, control, signals, and systems, there has not been any tutorial paper on this very important tool in this Transactions. Furthermore, we have found that there is no coverage on Gröbner bases at all in popular advanced engineering mathematics books which have been widely taught

Manuscript received August 3, 2007; revised November 14, 2007. This work was supported in part by JSPS.KAKENHI 19560448. This paper was recommended by Editor-in-Chief S. Basu.

Z. Lin is with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798 (e-mail: ezplin@ntu.edu.sg).

L. Xu is with the Department of Electronics and Information Systems, Akita Prefectural University, Akita 015-0055 Japan (e-mail: xuli@akita-pu.ac.jp).

N. K. Bose is with the Department of Electrical Engineering, The Pennsylvania State University, University Park, PA 16802-2705 USA (e-mail: bkn@engr.psu.edu).

Digital Object Identifier 10.1109/TCSI.2007.914007

to engineering students and used by engineers [2], [8], [56], [90]. The second objective is to give a reasonably detailed review of the applications of Gröbner bases in several topics related to multivariate (n-D)[1] polynomial matrix factorizations, which are fundamental to any n-D circuits, control, signals, and systems. The main reason for focusing on these topics is that n-D polynomial matrix factorization is one of the areas where the applications of Gröbner bases have helped to make a major breakthrough. Moreover, despite significant progress in this important area in recent years, there has been no review paper except for a book chapter published by one of the authors in 2003 [12]. The third objective is to document and analyze, albeit briefly because of space constraints, the more recent key papers (with attention to references therein of associated previous publications) on the applications of Gröbner bases to the diverse areas of circuits, control, signals, and systems, extending also to topics like algebraic coding theory and robotics, where electrical engineers have also played very important roles. To the best of our knowledge, we are unaware of any previous paper with a compilation of so many recent results on the applications of Gröbner bases in circuits, control, signals, and systems, and we believe this could make it a valuable reference for electrical engineers to continue and expand on research in this topical field.

The organization of this paper is as follows. In Section II, a brief tutorial on Gröbner bases is given to facilitate reading and comprehension of subsequent material and to introduce Gröbner bases to the reader as a powerful tool for relevant future applications. For more than a working knowledge of the subject, the serious reader is advised to consult any of the readable texts and expositions [1], [6], [12], [18], [19], [31], [33], [44], [114], [122], [129]. Another readable text that links Gröbner bases to related techniques involving resultant-subresultants and semialgebraic geometry is [77]. Section III contains a reasonably detailed review of several fundamental theoretical issues that occur in the use of Gröbner bases in n-D signals and systems. These topics include the primeness of n-D polynomial matrices, n-D unimodular polynomial matrix completion, and prime factorization of n-D polynomial matrices. This section contains some important recent theoretical results, with links to past seminal results. Section IV is devoted to a brief review of the applications of Gröbner bases to several other important areas of n-D signals and systems, which over the last three decades have matured into a subject matter with wide-ranging applications. Section V contains a summary of important one-dimensional (1-D) specializations, realizing that a significant number of readers are likely to be interested in at least a subset of such areas. Section VI is devoted to concluding remarks that highlight what could be achieved in the future in the context of developments presented in the earlier sections.

II. Tutorial on Gröbner Bases

Here, a brief tutorial on Gröbner bases is presented focusing on the most fundamental concepts, properties and computations. We first require some notation and basic definitions that will also be used throughout the paper.

[1]With a slight abuse of notation, we also use the term "n-D" to abbreviate "multivariate" or "n-variate." This usage is common among researchers in n-D signals and systems [10], [39].

Let K be an arbitrary but fixed field, $R = K[\mathbf{z}] = K[z_1, \ldots, z_n]$ be the polynomial ring in variables $z_1, \ldots, z_n$ over K, and $R^{\ell \times m}$ represents the class of $\ell \times m$ matrices with entries in R. We also write $R^{1\times m}$ as R^m, which represents the class of row vectors over R with m components. Note that the argument $[\mathbf{z}]$ is sometimes omitted to save space.

Definition 1: Let $f_1, \ldots, f_m \in K[\mathbf{z}]$ and $F = \{f_i | i = 1, \ldots, m\}$. The ideal generated by F is defined as

$$\text{Ideal}(F) = \left\{ \sum_{i=1}^{m} h_i f_i | h_i \in K[\mathbf{z}], i = 1, \ldots, m \right\}.$$

Definition 2: A module over the ring R, also called the R-module, is a commutative (Abelian) group M, usually written additively, together with a map $(r, m) \to rm$ from $R \times M$ to M satisfying the following conditions:

1) $r(m_1 + m_2) = rm_1 + rm_2$;
2) $(r_1 + r_2)m = r_1 m + r_2 m$;
3) $(r_1 r_2)m = r_1(r_2 m)$;
4) $1m = m$.

Definition 3: A free R-module is a module that is isomorphic to a direct sum of copies of R.

Definition 4: A submodule of an R-module M is a subset of M which is an R-module in its own right.

A typical example of a submodule is the submodule generated by a finite number of row vectors defined in the following.

Definition 5: Let $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_m \in K^r[\mathbf{z}]$ and $F = \{\boldsymbol{f}_i | i = 1, \ldots, m\}$. The submodule generated by F is defined as

$$\text{Module}(F) = \left\{ \sum_{i=1}^{m} h_i \boldsymbol{f}_i | h_i \in K[\mathbf{z}], i = 1, \ldots, m \right\}.$$

More notation and definitions will be introduced subsequently when they are required.

A. Why Gröbner Bases?

Many engineering applications require solving the following system of n-D polynomial equations, called polynomial system in short [107]:

$$f_1(\mathbf{z}) = 0; \cdots ; f_m(\mathbf{z}) = 0 \tag{1}$$

where $f_i(\mathbf{z}) \in \mathbf{C}[\mathbf{z}]$, $i = 1, \ldots, m$, are n-D polynomials in $z_1, \ldots, z_n$. Note that the coefficient field of complex numbers, denoted by $\mathbf{C}$, is assumed in this subsection for convenience of exposition, but the results are applicable to other algebraically closed coefficient fields. In the 1-D case ($n = 1$), the above polynomial system can be easily solved by utilizing the Euclidean Division Algorithm to find the greatest common divisor (g.c.d.) $d(z_1)$ of $f_1(z_1), \ldots, f_m(z_1)$, that is, $f_1(z_1) = 0, \ldots, f_m(z_1) = 0$ has a solution at $z_1 = z_{10} \in \mathbf{C}$ if and only if (iff) $(z_1 - z_{10})$ is a divisor of $d(z_1)$. In the n-D case ($n > 1$), the situation becomes much more complicated. Polynomial system (1) may still have a solution even if $f_1(\mathbf{z}), \ldots, f_m(\mathbf{z})$ do not have any nontrivial common divisor. Although polynomial systems occur in applications in almost all branches of engineering, as evident from the subsequent sections, in the engineering mathematics literature (e.g., [2] and [8]), they are often treated just as systems of nonlinear equations and numerical methods such as Newton's method are commonly adopted by practicing engineers. However, as pointed out in [73], "due to rounding errors, these numerical methods are unstable in unpredictable ways and usually do not find all solutions and frequently have problems with overdetermined systems." Moreover, numerical methods have the drawback of having to provide good initial solutions and are not applicable when parameterized solutions are desired. For polynomial systems with few indeterminates and of low polynomial degrees, the simple variable substitution method is commonly adopted by practicing engineers [2, p. 331]. The resultant-based method is a more general approach for solving polynomial systems and, in principle, is capable of enumerating all of the solutions, real or complex, when the considered polynomial system has a finite number of solutions [77]. A brief outline of this resultant-based method is given below. For clarity of presentation, here we only consider the polynomial system (1) specialized to the following 2-D case and refer the reader to [77] for the general case:

$$f_1(z_1, z_2) = 0; \quad f_2(z_1, z_2) = 0 \tag{2}$$

where $f_1(z_1, z_2)$ and $f_2(z_1, z_2)$ are assumed to be coprime (a common divisor, if present, can always be extracted first [9]). Express $f_1(z_1, z_2)$ and $f_2(z_1, z_2)$ as 1-D polynomials with z_2 as the main variable to yield

$$\begin{aligned} f_1(z_1, z_2) &= \sum_{i=0}^{\ell} a_i(z_1) z_2^i \\ f_2(z_1, z_2) &= \sum_{i=0}^{k} b_i(z_1) z_2^i \end{aligned} \tag{3}$$

where $a_\ell(z_1) \not\equiv 0$ and $b_k(z_1) \not\equiv 0$. The resultant $r_1(z_1)$ is the determinant of the Sylvester-type matrix given by [9], [77]:

$$r_1(z_1) = \det \begin{bmatrix} a_\ell(z_1) & \cdots & a_0(z_1) & & \\ & \ddots & \ddots & \ddots & \\ & & a_\ell(z_1) & \cdots & a_0(z_1) \\ b_k(z_1) & \cdots & b_0(z_1) & & \\ & \ddots & \ddots & \ddots & \\ & & b_k(z_1) & \cdots & b_0(z_1) \end{bmatrix} \begin{matrix} \left.\right\} k \text{ rows} \\ \\ \left.\right\} \ell \text{ rows} \end{matrix}.$$

The resultant $r_2(z_2)$ can be similarly defined and obtained by taking z_1 as the main variable.

The assumption of coprimeness of $f_1(z_1, z_2)$ and $f_2(z_1, z_2)$ ensures that $r_1(z_1) \not\equiv 0$, $r_2(z_2) \not\equiv 0$. It is well known [77] that common zeros of $f_1(z_1, z_2)$ and $f_2(z_1, z_2)$ can only come from the zeros of $r_1(z_1)$ and $r_2(z_2)$. It is then possible to obtain within arbitrary accuracy the finite set of pairs comprised of all possible combinations of zeros—one of $r_1(z_1)$ and the other of $r_2(z_2)$. However, candidate solutions to (2) obtained in this way are not necessarily the valid solutions due to the existence of spurious solutions by the resultant-based method. Hence, it is necessary to substitute each such pair to check whether or not it is a point on $f_1(z_1, z_2) = 0$, $f_1(z_1, z_2) = 0$, by polynomial evaluation. There are expedient and algorithmic ways to improve the checking of candidate solutions. The following illustrative example is taken and modified from [19].

Example 1: Find all solutions of the following polynomial system, where the two polynomials have been tested for coprimeness:

$$\begin{aligned} f_1(z_1, z_2) &= -3 + 2z_1 z_2 + z_1^2 z_2 = 0 \\ f_2(z_1, z_2) &= z_1^2 + z_1 z_2^2 = 0. \end{aligned} \tag{4}$$

The above polynomial system is rewritten as

$$\begin{aligned} f_1(z_1, z_2) &= \left(z_1^2 + 2z_1\right) z_2 - 3 = 0 \\ f_2(z_1, z_2) &= z_1 z_2^2 + 0 z_2 + z_1^2 = 0. \end{aligned} \tag{5}$$

The resultant $r_1(z_1)$ is the determinant of the Sylvester-type matrix

$$r_1(z_1) = \det \begin{bmatrix} z_1^2 + 2z_1 & -3 & 0 \\ 0 & z_1^2 + 2z_1 & -3 \\ z_1 & 0 & z_1^2 \end{bmatrix} \begin{matrix} \left.\right\} 2 \text{ rows} \\ \\ \left.\right\} 1 \text{ row} \end{matrix}$$

whose roots are the zeros of the univariate polynomial

$$r_1(z_1) = z_1 f_0(z_1) = z_1 \left(9 + 4z_1^3 + 4z_1^4 + z_1^5\right) = 0. \tag{6}$$

Similarly, we can obtain $r_2(z_2) = -3z_2^5 + 6z_2^3 + 9$, which has five zeros. Although $z_1 = 0$ is a solution to (6), it is not a feasible solution to (4) for any value of z_2 since $f_1(0, z_2) = -3 \neq 0$. Other candidate solutions to (4) consist of all of the possible combinations of pairing the five zeros of $f_0(z_1)$ with the five zeros of $r_2(z_2)$, which is a total of 25 candidate solutions. Instead of polynomial evaluation at each candidate solution, the remaining feasible solutions can be obtained more directly. Suppose that $z_1 = z_{10}$ is a solution to $f_0(z_1) = 0$. Substitute this value into both $f_1(z_1, z_2)$ and $f_2(z_1, z_2)$ and then extract the g.c.d. of the resulting univariate polynomials $f_1(z_{10}, z_2)$ and $f_2(z_{10}, z_2)$. Each z_2-zero of this g.c.d. in conjunction with z_{10} is a common zero of the original two polynomials. Subresultant theory can take care of multiple zeros, if present, as well. To save space, the detailed calculations are omitted.

To solve polynomial systems satisfactorily with parameterization of some coefficients (symbolically) when desired, a more effective approach is required. Gröbner bases are capable of finding exactly all of the solutions to a polynomial system and possess many other nice properties, as will be discussed next.

B. Gröbner Bases of Polynomial Ideals

To define Gröbner bases, we need to introduce an *admissible* term order $<_T$ for power products or monomials over $K[\mathbf{z}]$ [19], [68]. Note that monomial and power product are used interchangeably for elements in a ring like $K[\mathbf{z}]$. However, a monomial (vector) replaces the notion of power product in a module of column vectors of same size each with coordinates in $K[\mathbf{z}]$ [1, p.141]). Two commonly used term orders are the lexicographic order and the total degree lexicographic order. Without loss of generality, assume that the variables are ordered so that z_1 has the lowest order (ranking) leading up to z_n that has the highest order. Then, for the 2-D case, the lexicographic order is given by $1 <_T z_1 <_T z_1^2 <_T \cdots <_T z_2 <_T z_1 z_2 <_T z_1^2 z_2 <_T \cdots$, and the total degree lexicographic order is given by $1 <_T z_1 <_T z_2 <_T z_1^2 <_T z_1z_2 <_T z_2^2 <_T z_1^3 <_T \cdots$.

The following notations are defined with respect to (w.r.t.) a chosen term order $<_T$:

$\mathrm{cf}(f,t)$	coefficient of power product t in $f \in K[\mathbf{z}]$;
$\mathrm{lpp}(f)$	leading power product, i.e., the maximal power product with non-zero coefficient in $f \in K[\mathbf{z}]$ w.r.t. $<_T$;
$\mathrm{lcf}(f)$	leading coefficient, i.e., the coefficient of the $\mathrm{lpp}(f)$.

Let $f, g, h \in K[\mathbf{z}]$, $g \neq 0$. Then, h is called a *reduction* of f w.r.t. g, denoted by $f \to_g h$, iff there exist $b \in K$ and a power product u such that $\mathrm{cf}(f, u \cdot \mathrm{lpp}(g)) \neq 0$ and

$$h = f - b \cdot u \cdot g \tag{7}$$

where $b = \mathrm{cf}(f, u \cdot \mathrm{lpp}(g))/\mathrm{lcf}(g)$. Let $F \subseteq K[\mathbf{z}]$. Then, h is called a reduction of f modulo F, denoted by $f \to_F h$, iff there exists some $g \in F$ such that $f \to_g h$ [17], [19].

$h \in K[\mathbf{z}]$ is called in *normal form* (or in *reduced form*, or *a remainder*) modulo $F = \{f_1, \cdots, f_m\}$ iff there is no $h' \in K[\mathbf{z}]$ such that $h \to_F h'$. Further, h is called a normal form of $f \in K[\mathbf{z}]$ modulo F iff there is a sequence of reductions

$$f = k_0 \to_F k_1 \to_F k_2 \to_F \cdots \to_F k_q = h \tag{8}$$

and h is in normal form modulo F. Moreover, let NF be an algorithmic function that, for each F and f, produces an h that is a normal form of f modulo F, i.e., $\mathrm{NF}(F, f) = h$.

As a result of the above reduction sequence, h can be finally expressed in the form

$$h = f - \sum_{i=1}^{m} c_i f_i \tag{9}$$

where $c_i \in K[\mathbf{z}]$, $i = 1, \ldots, m$, are called the *cofactors* in the representation of h from f modulo F [17], [19].

Example 2: Let $f = 3z_1^2z_2 + 2z_1z_2^2 + z_1^3z_2^2$ and $F = \{f_1, f_2\}$, $f_1 = 3 + z_1 + z_1z_2 + z_1^3$, $f_2 = 4z_1z_2 + z_1z_2^2$ with the total degree lexicographic order. It is easy to see that $\mathrm{cf}(f, z_1^2z_2) = 3$, $\mathrm{cf}(f, z_1z_2^2) = 2$, $\mathrm{cf}(f, z_1^3z_2^2) = 1$, and $\mathrm{lpp}(f) = z_1^3z_2^2$, $\mathrm{lcf}(f) = 1$ due to $z_1^2z_2 <_T z_1z_2^2 <_T z_1^3z_2^2$. Similarly, we have that $\mathrm{lpp}(f_1) = z_1^3$, $\mathrm{lpp}(f_2) = z_1z_2^2$, $\mathrm{lcf}(f_1) = \mathrm{lcf}(f_2) = 1$.

Then, we have the sequence of reductions

$$f \to_{f_1} k_1 \to_{f_2} k_2 \to_{f_2} h \tag{10}$$

where

$$\begin{aligned} k_1 &= f - z_2^2 f_1 = -3z_2^2 + 3z_1^2z_2 + z_1z_2^2 - z_1z_2^3 \\ k_2 &= k_1 + z_2 f_2 = -3z_2^2 + 3z_1^2z_2 + 5z_1z_2^2 \\ h &= k_2 - 5f_2 = -20z_1z_2 - 3z_2^2 + 3z_1^2z_2. \end{aligned}$$

Since no further reduction of h modulo F is possible, $\mathrm{NF}(F, f) = h$. Substituting k_1 and k_2 into h yields

$$h = f - c_1 f_1 - c_2 f_2 \tag{11}$$

where $c_1 = z_2^2$ and $c_2 = 5 - z_2$ are the associated cofactors.

For a different sequence of reductions, another normal form of f modulo F is $h' = f + 4z_2f_1 - (6 + z_1^2)f_2 = 12z_2 - 20z_1z_2 + 3z_1^2z_2$.

This example shows that the normal forms for arbitrarily given f and F are not unique in general, which is substantially different from the 1-D case [19], [31]. It is desirable to have a set G of n-D polynomials such that $\mathrm{Ideal}(G) = \mathrm{Ideal}(F)$ and $\mathrm{NF}(G, f)$ can be uniquely determined. Such a G is just the Gröbner basis introduced by Buchberger [15]–[17], [19].

Definition 6: [17], [19] A subset G of $K[\mathbf{z}]$ is called a Gröbner basis of $\mathrm{Ideal}(G)$ (w.r.t. the term order chosen) iff any $f \in K[\mathbf{z}]$ has a unique normal form modulo G or, equivalently, for any $f \in \mathrm{Ideal}(G)$, $\mathrm{NF}(G, f) = 0$.

In addition, G is called a reduced Gröbner basis iff any $g \in G$ is monic, i.e., $\mathrm{lcf}(g) = 1$, and is in normal form modulo $(G - \{g\})$.

Buchberger's algorithm for construction of Gröbner basis terminates in a finite number of steps. Example 3 below illustrates the implementation of the algorithm, and its reading will be facilitated by arguments justifying its termination. Basically, each time a new polynomial is added to a specified set of polynomials, whose Gröbner basis G is under construction, it is in normal form with respect to the existing polynomials in G. A sequence of leading power products (alternately called head term) of each polynomial is obtained with the property that, except for the first few leading power products belonging to the initial polynomials, the succeeding term of the sequence is not a multiple of the previous terms. Buchberger showed that such a sequence (M-sequence) is finite [16]. In fact, this finiteness condition is quite easy to prove by induction.

To verify whether a given set $F \subset K[\mathbf{z}]$ is a Gröbner basis and, in the case that F is not a Gröbner basis, to transform F into a Gröbner basis G with $\mathrm{Ideal}(G) = \mathrm{Ideal}(F)$, we need the important concept of S-polynomial [15]–[17], [19].

The S-polynomial corresponding to $f_1, f_2 \in K[\mathbf{z}]$, denoted by $\mathrm{Sp}(f_1, f_2)$, is defined by

$$\mathrm{Sp}(f_1, f_2) = u_1 \cdot f_1 - \frac{\mathrm{lcf}(f_1)}{\mathrm{lcf}(f_2)} \cdot u_2 \cdot f_2 \tag{12}$$

where u_1 and u_2 are monomials such that $\mathrm{lcm}(\mathrm{lpp}(f_1), \mathrm{lpp}(f_2)) = u_1 \cdot \mathrm{lpp}(f_1) = u_2 \cdot \mathrm{lpp}(f_2)$, where lcm stands for the least common multiple.

Theorem 1: [17] A finite subset $G = \{g_1, \ldots, g_s\}$ of $K[\mathbf{z}]$ is a Gröbner basis iff, for all $g_i, g_j \in G$, $1 \leq i < j \leq s$, $\mathrm{NF}(G, \mathrm{Sp}(g_i, g_j)) = 0$.

Example 3: Consider $F = \{f_1, f_2\}$ given in Example 2. It is ready to have

$$\begin{aligned} \mathrm{Sp}(f_1, f_2) &= z_2^2 f_1 - z_1^2 f_2 \\ &= 3z_2^2 + z_1z_2^2 - 4z_1^3z_2 + z_1z_2^3 \\ \mathrm{NF}\,(F, \mathrm{Sp}(f_1, f_2)) &= \mathrm{Sp}(f_1, f_2) + 4z_2f_1 - (z_2 + 1)f_2 \\ &= 12z_2 + 3z_2^2 \\ &\neq 0. \end{aligned}$$

By Theorem 1, F is not a Gröbner basis, but it can be transformed into a Gröbner basis as follows (see, e.g., [17] and [19]). Adding (the monic version of $\mathrm{NF}(F, \mathrm{Sp}(f_1, f_2))$)$f_3 \triangleq$

$4z_2 + z_2^2$ to F, we get a new set $\tilde{F}_1 = \{f_1, f_2, f_3\}$ with $\mathrm{NF}(\tilde{F}_1, \mathrm{Sp}(f_1, f_2)) = 0$.

It is trivial to see that $f_2 = z_1 f_3$, i.e., f_2 can be reduced to zero w.r.t. f_3, and thus we can remove f_2 from $\tilde{F}_1$ to get $G \triangleq \{f_1, f_3\}$. It can now be verified that

$$\begin{aligned}\mathrm{Sp}(f_1, f_3) &= z_2^2 f_1 - z_1^3 f_3\\ &= 3z_2^2 + z_1 z_2^2 - 4z_1^3 z_2 + z_1 z_2^3\\ \mathrm{NF}\left(G_1, \mathrm{Sp}(f_1, f_3)\right) &= \mathrm{Sp}(f_1, f_3) + 4z_2 f_1 - (3 + z_1 + z_1 z_2) f_3\\ &= 0.\end{aligned}$$

Therefore, we have obtained the Gröbner basis $G = \{f_1, f_3\}$ with $\mathrm{Ideal}(G) = \mathrm{Ideal}(F)$.

The uniqueness property of normal form entails many important properties of Gröbner bases which play a crucial role in obtaining algorithmic solutions to numerous fundamental algebraic problems (see, e.g., [17]–[19]). One of the most important properties of Gröbner bases is the so-called elimination property w.r.t. the lexicographic ordering, which provides a systematical method for solving polynomial system (1) completely. This property is stated in the following theorem.

Theorem 2: [17], [68], [119] Let G be a Gröbner basis w.r.t. the lexicographic order of power products. Assume without loss of generality that $z_1 <_T \cdots <_T z_n$. Then

$$\mathrm{Ideal}(G) \cap K[z_1, \ldots, z_i] = \mathrm{Ideal}\left(G \cap K[z_1, \ldots, z_i]\right), \quad i = 1, \ldots, n \quad (13)$$

where the ideal on the right-hand side is formed in $K[z_1, \ldots, z_i]$.

This result means that the ith *elimination ideal* of G is generated by just those polynomials in G that depend only on the variables $z_1, \ldots, z_i$. Next, consider how the preceding theorem applies to the solving of polynomial system (1). For simplicity, assume that there are only a finite number of solutions to (1), i.e., the ideal generated by $F = \{f_1, \ldots, f_m\}$ is zero-dimensional. (For the case where the number of solutions is infinite, see, e.g., [19] and [66].) By Theorem 2, the Gröbner basis of F with respect to the lexicographic term ordering $z_1 <_T \ldots <_T z_n$ will have the form $G = \{g_1(z_1), g_{21}(z_1, z_2), \ldots, g_{2i_2}(z_1, z_2), g_{31}(z_1, z_2, z_3), \ldots, g_{3i_3}(z_1, z_2, z_3), \ldots, g_{n1}(z_1, \ldots, z_n), \ldots, g_{ni_n}(z_1, \ldots, z_n)\}$, where $i_2, \ldots, i_n$ are certain integers. Therefore, by the elimination property, the solutions to the polynomial system (1) can be obtained by solving, recursively, the triangular system of equations $g_1(z_1) = 0, g_{21}(z_1, z_2) = 0, \ldots, g_{ni_n}(z_1, \ldots, z_n) = 0$ by a sequence of forward substitutions followed by root-finding of the corresponding univariate polynomial. Note that there is no guarantee for exactly one polynomial involving the variables $z_1, \ldots, z_i$ representing the ith elimination ideal (see, e.g., [13, Example 2.5],). Unlike the resultant-based method discussed earlier, the fact that G is a Gröbner basis guarantees that any solution, say $z_1 = z_{10}$, obtained from $g_1(z_1) = 0$ is also part of the solution to the original polynomial system (1). Similarly, any solution, say $z_1 = z_{10}$, $z_2 = z_{20}$, obtained further from $g_{21}(z_{10}, z_2) = 0, \ldots, g_{2i_2}(z_{10}, z_2) = 0$ is again part of the solution to the original polynomial system (1) and so on. In this way, *all* of the solutions to polynomial system (1) are obtained without further checking the validity of each solution.

Now return to Example 1. The Gröbner basis for the ideal generated by $f_1(z_1, z_2)$ and $f_2(z_1, z_2)$ consists of the polynomials [19]

$$\begin{aligned}g_1(z_1) &= 9 + 4z_1^3 + 4z_1^4 + z_1^5\\ g_2(z_1, z_2) &= \frac{2}{3}z_1^2 + \frac{1}{3}z_1^3 + z_2.\end{aligned} \quad (14)$$

Note that $g_1(z_1)$ is identical to $f_0(z_1)$ in (6) and the spurious solution of $z_1 = 0$ does not exist here. Since $g_2(z_1, z_2)$ is linear in z_2, for any solution z_{10} to $g_1(z_1) = 0$, $g_2(z_{10}, z_2) = 0$ will produce one and only one solution (z_{10}, z_{20}), which is also a solution to the original polynomial system (4). There is no need for checking the validity of (z_{10}, z_{20}) here since any solution obtained by setting $g_1(z_1) = 0$ and $g_2(z_1, z_2) = 0$ is guaranteed to be a solution to (4).

We next move on to the tutorial on Gröbner bases of modules over a polynomial ring, which is a natural generalization of Gröbner bases of polynomial ideals.

C. Gröbner Bases of Modules Over a Polynomial Ring

In many engineering problems, particularly those in n-D signals and systems, it is also often required to solve the system of homogeneous linear equations with polynomial coefficients

$$x_1 \boldsymbol{f}_1 + \cdots + x_m \boldsymbol{f}_m = 0, \quad (15)$$

or, more generally, the inhomogeneous one

$$x_1 \boldsymbol{f}_1 + \cdots + x_m \boldsymbol{f}_m = \boldsymbol{f}_0, \quad (16)$$

where $\boldsymbol{f}_i, \boldsymbol{f}_0 \in K^r[\mathbf{z}]$ are given and $x_i \in K[\mathbf{z}]$ are to be found, $i = 1, \ldots, m$.

A vector $\boldsymbol{x} = [x_1 \cdots x_m] \in K^m[\mathbf{z}]$ whose entries satisfy (15) is called a syzygy of $\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_m\}$ or $F \triangleq [\boldsymbol{f}_1^T \cdots \boldsymbol{f}_m^T]^T$. (Note that, for convenience of description, some definitions and notations will be used for vectors of polynomials or polynomial tuples if they have been introduced for the set of all entries of the vectors.) The set of all such syzygies constitutes a submodule of $K^m[\mathbf{z}]$, called the syzygy module of F and denoted by $\mathrm{Syz}(\boldsymbol{f}_1, \ldots, \boldsymbol{f}_m)$ or $\mathrm{Syz}(F)$. To describe all syzygies of F, it is desirable to have a basis $\boldsymbol{h}_1, \ldots, \boldsymbol{h}_s \in K^m[\mathbf{z}]$ of $\mathrm{Syz}(F)$ that satisfies the conditions [1]: $\boldsymbol{h}_j F = \mathbf{0}$, $j = 1, \ldots, s$ and, for any $\boldsymbol{w} \in \mathrm{Syz}(F)$, there exist $v_1, \ldots, v_s \in K[\mathbf{z}]$ such that $\boldsymbol{w} = \sum_{j=1}^{s} v_j \boldsymbol{h}_j$.

To solve the inhomogeneous equation (16), it suffices to find a particular solution of (16) which is related to the submodule membership problem and to construct a basis for $\mathrm{Syz}(F)$.

These problems can be effectively solved by applying Gröbner bases of modules over $K[\mathbf{z}]$, which is briefly reviewed below [42], [135].

To define an admissible order on the r-tuples of power products, we only need to define an admissible order on a subset P of r-tuples of power products which consists of the tuples with only one nonzero component [42], [79], i.e.,

$$P \triangleq \left\{\left(0, \ldots, z_1^{i_1} \cdots z_n^{i_n}, \ldots, 0\right) | i_1, \ldots, i_n \in Z_+\right\}.$$

A partial order $<_M$ on P is defined by

$$(\forall \boldsymbol{p}_1, \boldsymbol{p}_2 \in P)\,[\boldsymbol{p}_1 <_M \boldsymbol{p}_2 \Leftrightarrow ((\exists q \neq 1, q \text{ power product})\boldsymbol{p}_2 = q \cdot \boldsymbol{p}_1)].$$

An admissible order $<_{M(T)}$ on P is defined to be a total order that satisfies the following properties [123]:

1) $(\forall \boldsymbol{p}_1, \boldsymbol{p}_2 \in P)[\boldsymbol{p}_1 <_M \boldsymbol{p}_2 \Rightarrow \boldsymbol{p}_1 <_{M(T)} \boldsymbol{p}_2]$;
2) $(\forall \boldsymbol{p}_1, \boldsymbol{p}_2 \in P)[\boldsymbol{p}_1 <_{M(T)} \boldsymbol{p}_2 \Rightarrow ((\forall q, q \text{ power product}) q \cdot \boldsymbol{p}_1 <_{M(T)} q \cdot \boldsymbol{p}_2)]$.

Let $<_T$ be an admissible order on the power products of $K[\mathbf{z}]$, for example, the lexicographic order or the total degree lexicographic order. Let $\boldsymbol{p} = (0, \ldots, p_i, \ldots, 0)$ and $\boldsymbol{q} = (0, \ldots, q_j, \ldots, 0) \in P$, where $p_i \neq 0$ occurs at the ith position of $\boldsymbol{p}$ and $q_j \neq 0$ at the jth position of $\boldsymbol{q}$. The *term first order based on* $<_T$ [123], or *highest order smallest-suffix order* [42], is such an example which determines the order $<_{M(T)}$ on P by comparing first p_i and q_j w.r.t. $<_T$, i.e.,

$$\boldsymbol{p} <_{M(T)} \boldsymbol{q} \Leftrightarrow [p_i <_T q_j \text{ or } (p_i = q_j \text{ and } i > j)].$$

Another possible one is the *index first order based on* $<_T$, which defines $<_{M(T)}$ on P by comparing first the indexes i and j, i.e.,

$$\boldsymbol{p} <_{M(T)} \boldsymbol{q} \Leftrightarrow [i > j \text{ or } (i = j \text{ and } p_i <_T q_j)].$$

For 2-tuples of 2-D power products, if $<_T$ is chosen to be the total degree lexicographic order, the term first order based on $<_T$ gives

$$[0\ 1] <_{M(T)} [1\ 0] <_{M(T)} [0\ z_1] <_{M(T)} [z_1\ 0] <_{M(T)} [0\ z_2] <_{M(T)} [z_2\ 0] <_{M(T)} \left[0\ z_1^2\right] <_{M(T)} \left[z_1^2\ 0\right] <_{M(T)} [0\ z_1z_2] <_{M(T)} [z_1z_2\ 0] <_{M(T)} \cdots$$

while the index first order based on $<_T$ gives

$$[0\ 1] <_{M(T)} [0\ z_1] <_{M(T)} [0\ z_2] <_{M(T)} \left[0\ z_1^2\right] <_{M(T)} [0\ z_1z_2] <_{M(T)} \cdots <_{M(T)} [1\ 0] <_{M(T)} [z_1\ 0] <_{M(T)} [z_2\ 0] <_{M(T)} \left[z_1^2\ 0\right] <_{M(T)} [z_1z_2\ 0] <_{M(T)} \cdots.$$

For a chosen admissible order $<_{M(T)}$, any nonzero $\boldsymbol{f} \in K^r[\mathbf{z}]$ can be uniquely expressed as

$$\boldsymbol{f} = \sum_{i=1}^{\sigma} \text{cf}(\boldsymbol{f}, \boldsymbol{p}_i) \cdot \boldsymbol{p}_i,\ \text{cf}(\boldsymbol{f}, \boldsymbol{p}_i) \in K \setminus \{0\}, \quad \boldsymbol{p}_i \in P, \boldsymbol{p}_1 <_{M(T)} \boldsymbol{p}_2 <_{M(T)} \cdots <_{M(T)} \boldsymbol{p}_\sigma. \tag{17}$$

where $\text{cf}(\boldsymbol{f}, \boldsymbol{p}_i)$ denotes the *coefficient of* $\boldsymbol{p}_i$ in $\boldsymbol{f}$ [42], [123].

Define also the following notations w.r.t. the chosen order:

- $\text{lppt}(\boldsymbol{f})$ *leading power product tuple of* $\boldsymbol{f}$, i.e., $\boldsymbol{p}_\sigma$;
- $\text{lpp}(\boldsymbol{f})$ *leading power product of* $\boldsymbol{f}$, i.e., the nonzero component of $\boldsymbol{p}_\sigma$;
- $\text{lcf}(\boldsymbol{f})$ *leading coefficient of* $\boldsymbol{f}$, i.e., $\text{cf}(\boldsymbol{f}, \boldsymbol{p}_\sigma)$;
- $\text{lt}(\boldsymbol{f})$ *leading term of* $\boldsymbol{f}$, i.e., $\text{lcf}(\boldsymbol{f}) \cdot \text{lpp}(\boldsymbol{f})$;
- $\text{hp}(\boldsymbol{f})$ *head position of* $\boldsymbol{f}$, i.e., if the nonzero component of $\boldsymbol{p}_\sigma$ occurs at the kth position, then $\text{hp}(\boldsymbol{f}) = k$.

Let $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} \in K^r[\mathbf{z}]$, $\boldsymbol{g} \neq \mathbf{0}$. Then, the *reduction* relation $\boldsymbol{f} \rightarrow_{\boldsymbol{g}}$ is defined as

$$\boldsymbol{f} \rightarrow_{\boldsymbol{g}} \boldsymbol{h} \Leftrightarrow (\exists v, v \text{ power product})\left[\text{cf}(\boldsymbol{f}, v \cdot \text{lppt}(\boldsymbol{g})) \neq 0 \text{ and } \boldsymbol{h} = \boldsymbol{f} - \frac{\text{cf}(\boldsymbol{f}, v \cdot \text{lppt}(\boldsymbol{g}))}{\text{lcf}(\boldsymbol{g})} \cdot v \cdot \boldsymbol{g}\right]. \tag{18}$$

Let $F \subseteq K^r[\mathbf{z}]$. Then, $\boldsymbol{h}$ is a reduction of $\boldsymbol{f}$ modulo F, denoted by $\boldsymbol{f} \rightarrow_F \boldsymbol{h}$, iff there exists $\boldsymbol{g} \in F$ such that $\boldsymbol{f} \rightarrow_{\boldsymbol{g}} \boldsymbol{h}$. Further, $\boldsymbol{h}$ is said to be in *normal form* (or, in *reduced form*, or *a remainder*) modulo F iff there is no $\boldsymbol{h}' \in K^r[\mathbf{z}]$ such that $\boldsymbol{h} \rightarrow_F \boldsymbol{h}'$. Then, $\boldsymbol{h}$ is a normal form of $\boldsymbol{f}$ modulo F iff there is a sequence of reductions such that

$$\boldsymbol{f} = \boldsymbol{k}_0 \rightarrow_F \boldsymbol{k}_1 \rightarrow_F \boldsymbol{k}_2 \rightarrow_F \cdots \rightarrow_F \boldsymbol{k}_q = \boldsymbol{h}$$

and $\boldsymbol{h}$ is in normal form modulo F. Furthermore, let NF be an algorithmic function that, for each F and $\boldsymbol{f}$, produces an $\boldsymbol{h}$ that is a normal form of $\boldsymbol{f}$ modulo F, i.e., $\text{NF}(F, \boldsymbol{f}) = \boldsymbol{h}$.

Definition 7: A finite subset G of $K^r[\mathbf{z}]$ is a Gröbner basis of $\text{Module}(G)$ (w.r.t. the order chosen) iff any $\boldsymbol{f} \in K^r[\mathbf{z}]$ has a unique normal form modulo G or, equivalently, for any $\boldsymbol{f} \in \text{Module}(G)$, $\text{NF}(G, \boldsymbol{f}) = 0$. In addition, G is called a reduced Gröbner basis iff, for all $\boldsymbol{g} \in G$, $\text{lcf}(\boldsymbol{g}) = 1$ and $\boldsymbol{g}$ is in normal form modulo $(G - \{\boldsymbol{g}\})$.

The notion of S-polynomial and the related properties can also be generalized [42], [79], [123].

The S-polynomial of $\boldsymbol{f}_1, \boldsymbol{f}_2 \in K^r[\mathbf{z}]$, is defined by

$$\text{Sp}(\boldsymbol{f}_1, \boldsymbol{f}_2) = \begin{cases} u_1 \cdot \boldsymbol{f}_1 - \frac{\text{lcf}(\boldsymbol{f}_1)}{\text{lcf}(\boldsymbol{f}_2)} \cdot u_2 \cdot \boldsymbol{f}_2, & \text{if hp}(\boldsymbol{f}_1) = \text{hp}(\boldsymbol{f}_2) \\ 0, & \text{otherwise} \end{cases}$$

where u_1 and u_2 are monomials such that $\text{lcm}(\text{lpp}(\boldsymbol{f}_1), \text{lpp}(\boldsymbol{f}_2)) = u_1 \cdot \text{lpp}(\boldsymbol{f}_1) = u_2 \cdot \text{lpp}(\boldsymbol{f}_2)$.

Theorem 3 [42], [123]: A finite subset $G = \{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_s\}$ of $K^r[\mathbf{z}]$ is a Gröbner basis iff, for all $\boldsymbol{g}_i, \boldsymbol{g}_j \in G$, $1 \leq i < j \leq s$, $\text{NF}(G, \text{Sp}(\boldsymbol{g}_i, \boldsymbol{g}_j)) = 0$.

Similar to the polynomial case, constructive algorithms can be established to test whether a given $F \subseteq K^r[\mathbf{z}]$ is a Gröbner basis, and, if it is not, to calculate the Gröbner basis G of F. See, e.g., [42], [123] for the related details as well as the solution problem of (16). In the following, we shall concentrate on the important topic of the construction of a basis of $\text{Syz}(F)$ using Gröbner bases.

First, we have the following result for a basis of the syzygy module for a Gröbner basis [42], [123], [142].

Let $G = \{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_s\}$ be a Gröbner basis. For all $\boldsymbol{g}_i, \boldsymbol{g}_j$, $1 \leq i < j \leq s$, let $u_{ij}, v_{ij}, k_{ij}^1, \ldots, k_{ij}^s \in K[\mathbf{z}]$ be such that

$$\text{Sp}(\boldsymbol{g}_i, \boldsymbol{g}_j) = u_{ij}\boldsymbol{g}_i + v_{ij}\boldsymbol{g}_j = k_{ij}^1\boldsymbol{g}_1 + \cdots + k_{ij}^s\boldsymbol{g}_s \tag{19}$$

where $k_{ij}^1, \ldots, k_{ij}^s$ are extracted from the reduction of $\text{Sp}(\boldsymbol{g}_i, \boldsymbol{g}_j)$ to $\mathbf{0}$. Then

$$\left\{u_{ij}\boldsymbol{e}_i + v_{ij}\boldsymbol{e}_j - \left[k_{ij}^1 \cdots k_{ij}^s\right] | 1 \leq i < j \leq s\right\} \tag{20}$$

is a basis of $\text{Syz}(G)$, where $\boldsymbol{e}_i = [0 \cdots 1 \cdots 0]$ with only the ith entry being 1.

Denote the subsets $\{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_s\}$ and $\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_m\}$ of $K^r[\mathbf{z}]$ in the form of vector of r-tuples of polynomials, i.e., $G \triangleq$

$[\boldsymbol{g}_1^T \cdots \boldsymbol{g}_s^T]^T \in K^{s\times r}$ and $F \triangleq [\boldsymbol{f}_1^T \cdots \boldsymbol{f}_m^T]^T \in K^{m\times r}$. Let G be a Gröbner basis for F with $\mathrm{Module}(G) = \mathrm{Module}(F)$.

By reducing the $\boldsymbol{f}_i$'s to $\mathbf{0}$ modulo G and collecting the multiples of the $\boldsymbol{g}_j$'s used in the reduction, we can obtain a matrix $Y \in K^{m\times s}[\mathbf{z}]$ such that

$$F = YG \tag{21}$$

while, by collecting the multiples of $\boldsymbol{f}_j$'s when constructing the Gröbner basis G from F, we can obtain a matrix $X \in K^{s\times m}[\mathbf{z}]$ such that

$$G = XF. \tag{22}$$

Moreover, let the rows of S be a basis of $\mathrm{Syz}(G)$ obtained in (20). Then, it can be shown that the rows of the following matrix Q is a basis of $\mathrm{Syz}(F)$ [42], [123], [142]:

$$Q = \begin{bmatrix} I_m - YX \\ SX \end{bmatrix}. \tag{23}$$

The following example is taken and adapted from [1].

Example 4: Let $\boldsymbol{f}_1 = [z_2\ z_1 + z_2 + 2z_1^2]$, $\boldsymbol{f}_2 = [z_2\ z_1 - z_2]$, and $\boldsymbol{f}_3 = [z_2\ z_1 + z_1^2]$ with the index first order (based on the total degree lexicographic order $<_T$).

$\boldsymbol{f}_i$, $i = 1, 2, 3$, can be expressed as

$$\begin{aligned}\boldsymbol{f}_1 &= [0\ z_1] + [0\ z_2] + 2\left[0\ z_1^2\right] + [z_2\ 0]\\ \boldsymbol{f}_2 &= [0\ z_1] - [0\ z_2] + [z_2\ 0]\\ \boldsymbol{f}_3 &= [0\ z_1] + \left[0\ z_1^2\right] + [z_2\ 0]\end{aligned}$$

and thus we have $\mathrm{lppt}(\boldsymbol{f}_i) = [z_2\ 0]$, $\mathrm{lpp}(\boldsymbol{f}_i) = z_2$, $\mathrm{lcf}(\boldsymbol{f}_i) = 1$, $\mathrm{lt}(\boldsymbol{f}_i) = z_2$, and $\mathrm{hp}(\boldsymbol{f}_i) = 1$, $i = 1, 2, 3$. Further, as

$$\mathrm{Sp}(\boldsymbol{f}_1, \boldsymbol{f}_2) = f_1 - f_2 = \left[0\ 2z_2 + 2z_1^2\right]$$

is already in normal form, we introduce the monic version of it, i.e., $\boldsymbol{f}_4 \triangleq [0\ z_2 + z_1^2]$ to have the new set $\{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3, \boldsymbol{f}_4\}$. It is easy to see that

$$\begin{aligned}&\boldsymbol{f}_1 \to_{\boldsymbol{f}_3} \left[0\ z_2 + z_1^2\right] (= \boldsymbol{f}_1 - \boldsymbol{f}_3) \to_{\boldsymbol{f}_4} \mathbf{0},\\ &\boldsymbol{f}_2 \to_{\boldsymbol{f}_3} -\left[0\ z_2 + z_1^2\right] (= \boldsymbol{f}_2 - \boldsymbol{f}_3) \to_{\boldsymbol{f}_4} \mathbf{0}.\end{aligned}$$

Thus, we can remove $\boldsymbol{f}_1, \boldsymbol{f}_2$ to have just the set $\{\boldsymbol{f}_3, \boldsymbol{f}_4\}$. Since $\mathrm{hp}(\boldsymbol{f}_4) = 2$ is different from $\mathrm{hp}(\boldsymbol{f}_3) = 1$, $\mathrm{Sp}(\boldsymbol{f}_3, \boldsymbol{f}_4) = 0$, that is, we have obtained the reduced Gröbner basis $\{\boldsymbol{f}_3, \boldsymbol{f}_4\}$ for $\mathrm{Module}(\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3)$.

Next, consider the problem of finding a basis for $\mathrm{Syz}(\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3)$. Let $\boldsymbol{g}_1 = \boldsymbol{f}_4$, $\boldsymbol{g}_2 = \boldsymbol{f}_3$, and define $F = [\boldsymbol{f}_1^T\ \boldsymbol{f}_2^T\ \boldsymbol{f}_3^T]^T$ and $G = [\boldsymbol{g}_1^T\ \boldsymbol{g}_2^T]^T$. Then, as $\mathrm{Sp}(\boldsymbol{g}_1, \boldsymbol{g}_2) = \boldsymbol{g}_1 - \boldsymbol{g}_2$, we have that $u_{12} = k_{12}^1 = 1$, $v_{12} = k_{12}^2 = -1$, and the basis of $\mathrm{Syz}(G)$ is

$$\left\{u_{12}[1\ 0] + v_{12}[0\ 1] - \left[k_{12}^1\ \ k_{12}^2\right] = [0\ 0]\right\}$$

which leads to $S = [0\ 0]$. Also, it is easy to construct the relations $G = XF$ and $F = YG$ with

$$X = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then, by (23), we can obtain a 4×3 matrix Q which has in fact only one nonzero row: $[1\ 1\ {-2}]$ This gives a basis for $\mathrm{Syz}(F)$, and it is ready to verify that $[1\ 1\ \ {-2}]F = \mathbf{0}$.

With the above tutorial of Gröbner bases, we are now ready to review the applications of Gröbner bases in signals and systems. We begin in the next section with a reasonably detailed review of applications of Gröbner bases in n-D polynomial matrices in relation to several fundamental issues in n-D systems theory, which has diverse applications in various engineering problems including circuits, control, signals, and systems [9], [12].

III. Applications of Gröbner Bases in Multivariate Polynomial Matrices

The method of Gröbner bases has been used as a main computational tool in various problems arising in n-D signals and systems over not only n-D polynomials but also Laurent rings [98]. The topics we review in some detail in this section are the primeness of n-D polynomial matrices, n-D unimodular polynomial matrix completion, and factorizations for n-D polynomial matrices. The reason for focusing on n-D polynomial matrices is motivated by the fact that many n-D linear systems and various problems in circuits, control, systems, and signal processing can be described by n-D polynomial and matrix-fraction description of rational matrices with polynomial matrices. Following the classical approach of Youla and coworkers [140], [141], we adopt the language of polynomials as much as possible and use only a minimum of modern abstract algebra.

A. Primeness of n-D Polynomial Matrices

We first recall the following definition [126], [140].

Definition 8: Let $F \in R^{\ell\times m}$ with $\ell \le m$ and be of full rank. Then F is said to be:

1) zero left prime (ZLP) if all the $\ell\times\ell$ minors of F generate the unit ideal R;
2) minor left prime (MLP) if all of the $\ell \times \ell$ minors of F are relatively prime or coprime, i.e., the g.c.d. of all of the $\ell \times \ell$ minors of F, denoted by $\mathrm{d}(F)$, is a nonzero constant;
3) factor left prime (FLP) if in any polynomial matrix factorization $F = G_1F_1$ in which G_1 is a square matrix, G_1 is necessarily a unimodular matrix, i.e., $\det G_1$ is a nonzero constant in K.

Zero right prime (ZRP) and minor right prime (MRP), etc., can be similarly defined for matrices $F \in R^{\ell\times m}$ with $\ell \ge m$. For convenience of exposition, throughout the paper, we assume that $\ell \le m$ and only consider ZLP, MLP, and FLP. Moreover, due to space limitation, we only consider the case where F if of full row rank. For the case where F is not of full row rank, see [64] and [125].

Note that ZLP $\Rightarrow$ MLP $\Rightarrow$ FLP. When $n \ge 3$, these three concepts are pairwise different; when $n = 2$, ZLP is not equivalent to MLP, but MLP is the same as FLP; when $n = 1$, all three concepts coincide [140].

Note also that, in n-D signals and systems, it is often required to know whether two n-D polynomial matrices $A \in R^{\ell\times\ell}$ and $B \in R^{\ell\times r}$ are zero left coprime (ZLC), etc. This can be easily done by testing whether their associated matrix $F = [A\ B]$ is ZLP, etc.

When K is an algebraically closed field such as the field of complex numbers $\mathbf{C}$, F is ZLP iff all the $\ell \times \ell$ minors of F have no common zeros in K.

Now, we explore the application of Gröbner bases to the determination of various types of primeness of a given n-D polynomial matrix F.

Let $a_1, \ldots, a_\beta$ denote the $\ell \times \ell$ minors of the matrix $F \in R^{\ell \times m}$, where $\beta = \binom{m}{\ell} = (m!/(m-\ell)!\ell!)$. For F to be ZLP, a condition equivalent to that in Definition 8–1) is the existence of $e_1, \ldots, e_\beta \in R$ such that

$$\sum_{i=1}^{\beta} e_i a_i = 1 \tag{24}$$

that is, the multiplicative identity 1 is contained in the ideal formed by the polynomials $a_1, \ldots, a_\beta$. Using the membership property of Gröbner bases, we can easily determine whether (24) holds by simply calculating the Gröbner basis of the ideal formed by polynomials $a_1, \ldots, a_\beta$ and then checking whether this Gröbner basis contains the unit 1. If the answer is yes, F is ZLP. Otherwise, F is not ZLP.

For F to be MLP, a condition equivalent to that in Definition 8-2) is that, for each k, $k = 1, 2, \ldots, n$, there exist $e_{1,k}, \ldots, e_{\beta,k} \in R$ such that

$$\sum_{i=1}^{\beta} e_{i,k} a_i = \psi_k(\mathbf{z} \setminus z_k), \quad k = 1, 2, \ldots, n. \tag{25}$$

Again, this MLP property can be tested by using Gröbner bases. To do this, for each k, $k = 1, 2, \ldots, n$, obtain the Gröbner basis of the ideal formed by the polynomials $a_1, \ldots, a_\beta$ using lexicographic ordering, $z_t <_T z_k, \forall t \neq k$, and then check whether this Gröbner basis contains at least one element that is independent of z_k, i.e., $\psi_k(\mathbf{z} \setminus z_k)$. If the answer is yes for all $k = 1, 2, \ldots, n$, F is MLP. Otherwise, F is not MLP.

The next illustrative example is taken and adapted from [12].

Example 5: Let $F(\mathbf{z}) = \begin{bmatrix} z_1z_2 + 1 & -1 & 1 \\ z_2 & z_1z_2 - 1 & -z_1 \end{bmatrix}$. The objective here is to determine whether F is MLP. The set M of all 2 × 2 minors of F are

$$\begin{aligned} a_1(\mathbf{z}) &= z_1^2 z_2^2 + z_2 - 1 \\ a_2(\mathbf{z}) &= -z_1^2 z_2 - z_1 - z_2 \\ a_3(\mathbf{z}) &= -z_1 z_2 + z_1 + 1. \end{aligned}$$

Using the lexicographic order $z_2 <_T z_1$, the Gröbner basis $\tilde{G}_1$ of $\mathrm{Ideal}(M)$ is calculated to be

$$\tilde{G}_1 = \left\{ z_2^3 - 2z_2^2 + 3z_2 - 1, z_2^2 - z_2 + 2 + z_1 \right\}.$$

Alternatively, using the lexicographic order $z_1 <_T z_2$, the Gröbner basis $\tilde{G}_2$ of $\mathrm{Ideal}(M)$ is calculated to be

$$\tilde{G}_2 = \left\{ z_1^3 + 2z_1^2 + z_1 + 1, z_1^2 + 2z_1 + z_2 \right\}.$$

Thus, the polynomial $\psi_1(\mathbf{z} \setminus z_1)$ is $(z_2^3 - 2z_2^2 + 3z_2 - 1)$ and, similarly, the polynomial $\psi_2(\mathbf{z} \setminus z_2)$ is $(z_1^3 + 2z_1^2 + z_1 + 1)$. In addition, by keeping track of how each element of $\tilde{G}_i$, $i = 1, 2$ is generated, the following relations are obtained:

$$\begin{aligned} z_1^3 + 2z_1^2 + z_1 + 1 &= (0)a_1(\mathbf{z}) - z_1 a_2(\mathbf{z}) \\ &\quad + \left(z_1^2 + 1\right) a_3(\mathbf{z}) \end{aligned} \tag{26}$$

$$\begin{aligned} z_2^3 - 2z_2^2 + 3z_2 - 1 &= (0)a_1(\mathbf{z}) - (z_2 - 1)^2 a_2(\mathbf{z}) \\ &\quad + \left(z_1 z_2^2 - z_1 z_2 + 2z_2 - 1\right) a_3(\mathbf{z}). \end{aligned} \tag{27}$$

Since $n = 2$ in this example, it is then clear from (25)–(27) that F is MLP.

The determination of whether a given n-D polynomial matrix F is ZLP or MLP can also be formulated as the solvability problems of certain polynomial matrix equations. It has been known [140] that F is ZLP iff there exists $X \in R^{m \times \ell}$ such that

$$FX = I_\ell \tag{28}$$

and that F is MLP iff there exist $X_k \in R^{m \times \ell}$ and $\psi_k(\mathbf{z} \setminus z_k)$ such that

$$FX_k = \psi_k(\mathbf{z} \setminus z_k) I_\ell, \quad k = 1, \ldots, n. \tag{29}$$

The solvability of (28) and (29) can be easily reduced to the corresponding submodule membership problems and thus can be solved effectively by using Gröbner bases of modules over a polynomial ring described in Section II. Moreover, if the equations are solvable, it is also easy to construct the corresponding solutions, which are required in some applications, by using Gröbner bases of modules. See, e.g., [12] and [135] for more details on this topic.

The determination of whether a given n-D polynomial matrix F is FLP or not turns out to be much more difficult, either by conventional methods or by Gröbner bases. This was highlighted as one of the most important problems in n-D systems theory by Youla and Gnavi in [140]. A partial solution to this open problem, as well as to the related problem of factor prime factorizations, has recently been proposed by Wang in [127], using Gröbner bases. We will review this topic in Section III-C.

More detailed discussions on the primeness of n-D polynomial matrices can be found in [12], [39], [130], [143], and [144].

B. Completion of n-D Unimodular Polynomial Matrices

As Youla and Pickel rightly pointed out in [141], "some of the most impressive accomplishments in circuits and systems have been obtained by an in-depth exploitation of the properties of elementary polynomial matrices. Algorithms for the construction of such matrices are, therefore, of both theoretical and practical importance." In fact, one of the most important results in mathematics in the second half of the 20th century, the Quillen–Suslin theorem [57], [58], [104], [118], is closely related to the completion of a nonsquare n-D unimodular polynomial matrix into a square one. Note that Youla and Pickel referred to a square unimodular matrix as an elementary matrix in [141]. However, since the term "elementary matrix" is commonly reserved for a matrix associated with an elementary operation, we stick to the term "unimodular matrix" instead of "elementary matrix" in this paper.

The original Quillen–Suslin theorem was a nonconstructive solution to the famous Serre's problem (often referred to as Serre's Conjecture) on whether or not projective modules over polynomial rings are free (see the recent book on this subject by Lam [58]). The Quillen–Suslin theorem was proved constructively by Logar and Sturmfels [71] in 1992, Park and Woodburn [93] in 1995, using Gröbner bases, and more recently, by Lombardia and Yengui [72]. In [141], Youla and Pickel gave an elementary proof (constructive in most parts) of the Quillen-Suslin theorem, using mostly the language of polynomials. Since the n-D unimodular matrix completion problem is fundamental to n-D signals and systems, we briefly review the elementary approach to this problem by Youla and Pickel [141] and adopt the Gröbner bases methodology whenever possible. For convenience of exposition, in this subsection, we assume the coefficient field to be the field of complex numbers $\mathbf{C}$.

In [140] and [141], it was shown that an n-D polynomial matrix could be completed into a (square) unimodular polynomial matrix iff a unimodular row vector $F = [f_1, \ldots, f_m] \in \mathbf{C}^m[\mathbf{z}]$ could be completed into a unimodular matrix. A necessary condition is that $f_1, \ldots, f_m$ are zero coprime, and this is assumed in the following.

By Gröbner bases, we can construct $e_1, \ldots, e_\beta \in \mathbf{C}[\mathbf{z}]$ such that

$$\sum_{i=1}^{m} e_i f_i = 1. \tag{30}$$

Furthermore, it may be assumed that both e_1 and e_2 are nonzero polynomials; otherwise, a simple technique (see [141]) can be applied to make both e_1 and e_2 nonzero polynomials. The next step is to ensure that both f_1 and e_1 are monic polynomials in z_n. A very important result from the resultant theory shows that, with some simple preparations if necessary (see [141]), there exist $d_1, d_2 \in \mathbf{C}[\mathbf{z}]$, such that

$$d_2 e_1 - d_1 e_2 = q, \tag{31}$$

where $q \in \mathbf{C}[z_1, \ldots, z_{n-1}]$ and does not vanish for any prescribed $(n-1)$-tuple $(c_1, c_2, \ldots, c_{n-1})$ of constants. Note that d_1 and d_2 in (31) can be obtained by using Gröbner bases. Once d_1 and d_2 are constructed as in (31), the next step is to construct $L \in \mathbf{C}^{m \times m}[\mathbf{z}]$ such that

$$FL = [1, 0, \ldots, 0] \tag{32}$$

where $\det L = q$. With simple manipulation, F can be row-bordered up into a matrix $B \in \mathbf{C}^{m \times m}[\mathbf{z}]$ such that $\det B = q^{m-2}$. The next important step is to exploit the properties of q above and Hilbert's finite-basis theorem [33] by defining the set of admissible determinant functions $D[z_1, \ldots, z_{n-1}](F)$ for the given ZLP row vector F. Specifically, $g \in \mathbf{C}[z_1, \ldots, z_{n-1}]$ is said to be an admissible determinant function for F and to belong to $D[z_1, \ldots, z_{n-1}](F)$ if some $m \times m$ polynomial matrix B includes F in its first row and is such that $\det B = g^t$ for some nonnegative integer t. By constructing appropriate polynomials $g_1, g_2, \ldots \in D[z_1, \ldots, z_{n-1}](F)$ and forming ideals $I_1 = \{g_1\}, I_2 = \{g_1, g_2\}, \ldots$ with $I_1 \subset I_2 \subset \ldots$, it follows from Hilbert's finite-basis theorem [33] that the strictly ascending chain of the ideals must end after a finite number of steps p and the admissible determinant functions $g_1, g_2, \ldots, g_p$ are zero coprime, that is, $D[z_1, \ldots, z_{n-1}](F)$ contains an ideal that has the unit 1.

The final step in the proof of the Quillen–Suslin theorem by the approach of Youla and Pickel [141] is to show that the set $D[z_1, \ldots, z_{n-1}](F)$ of admissible determinant functions is itself an ideal in $\mathbf{C}[z_1, \ldots, z_{n-1}]$. The proof is constructive but is a rather long one and hence is omitted here. We refer the reader to [141] for more details. However, in our opinion, the constructive steps adopted by Youla and Pickel in showing that $D[z_1, \ldots, z_{n-1}](F)$ is an ideal do not lead to the construction of a unimodular matrix row-bordering F since only the existence of such a unimodular matrix was established in [141]. It is probably for this reason that only the work in [71] and [92] (and, more recently, [72]) are considered to be constructive proofs for the Quillen–Suslin theorem in the literature [58], [64]. Since the constructive proofs in [71], [72], and [92] are quite lengthy, we skip the descriptions here due to space limitation. The interested reader can refer to [71], [72], and [92] for more details.

It should be pointed out that Park proposed a very simple technique for the n-D unimodular polynomial matrix completion problem by computing a globally minimal set of generators for a syzygy module using Gröbner bases [92]. The technique is simple and computationally quite efficient but heuristic. It is not a constructive proof of the Serre conjecture since it does not always produce a globally minimal set of generators for a syzygy module even by using Gröbner bases.

C. *Prime Factorizations of n-D Polynomial Matrices*

The next important topic to be reviewed is the prime factorization of an n-D $(n > 2)$ polynomial matrix. Note that the prime factorization problem for the 2-D case has been completely solved over two decades ago [12], [45], [140]. There are various types of prime factorization schemes corresponding to the different types of primeness discussed in Section III-A, and the three most common ones are defined below.

Definition 9: Let $F \in R^{\ell \times m}$ and assume that $F = G_1 F_1$ with $G_1 \in R^{\ell \times \ell}$, and $F_1 \in R^{\ell \times m}$. We say that:

1) F admits a ZLP factorization if F_1 is ZLP;
2) F admits a MLP factorization if F_1 is MLP;
3) F admits a FLP factorization if F_1 is FLP.

We recall a useful concept introduced in [62] and [115].

Definition 10: [62], [115] Let $F \in R^{\ell \times m}$ and $a_1, \ldots, a_\beta$ be the $\ell \times \ell$ minors of the matrix F. Extracting the g.c.d. d of $a_1, \ldots, a_\beta$ gives

$$a_i = d b_i, \qquad i = 1, \ldots, \beta. \tag{33}$$

Then, $b_1, \ldots, b_\beta$ are called the *generating set* [62] or *reduced minors* [115] of F.

The following conjectures were raised in [64] (also see [63]).

Conjecture 1: [64] Let $F \in R^{\ell \times m}$. If $b_1, \ldots, b_\beta$ are zero coprime, there exists $E \in R^{(m-\ell) \times m}$ such that $U = [F^T \; E^T]^T \in R^{m \times m}$ with $\det U = d$.

Conjecture 2: [63], [64] Let $F \in R^{\ell \times m}$. If $b_1, \ldots, b_\beta$ are zero coprime, F can be factored as $F = G_1 F_1$, for some $F_1 \in R^{\ell \times m}$, $G_1 \in R^{\ell \times \ell}$ with $\det G_1 = d$.

The above conjectures, shown to be equivalent in [64], have drawn some attention from both the mathematics and engineering communities. In particular, Conjecture 1 can be considered as a kind of generalization of Serre's Conjecture (see [58] for more details on a comprehensive discussion on the development of Serre's Conjecture and related issues). Conjecture 2 is just the ZLP factorization problem defined in Definition 9 and has been proved to be true by Pommaret [102], Srinivas [117], Wang and Feng [125], and partially by Park [97] using different approaches. We briefly quote the following result from [102] and [125].

Theorem 4: [102], [125] Let $F \in R^{\ell \times m}$, $d = \mathrm{d}(F)$, and $b_1, \cdots, b_\beta$ be the reduced minors. Then the following statements are equivalent.

1) $b_1, \cdots, b_\beta$ generate the unit ideal R.
2) There exist polynomial matrices $G_1 \in R^{\ell \times \ell}$ and $F_1 \in R^{\ell \times m}$ such that $F = G_1 F_1$ with $\det G_1 = d$ and F_1 being ZLP.

Moreover, some interesting properties concerning ZLP factorizations in [125] and two related definitions in [126] are also recalled here.

Definition 11: [126] Let M be an R-module. The torsion submodule of M is defined as $\mathrm{Torsion}(M) = \{u \in M | \exists a \in R, au = 0, a \neq 0\}$. M is said to be torsion-free if $\mathrm{Torsion}(M) = 0$.

Definition 12: [126] Let N be a submodule of R^m, and $a \in R$. We define $N : a = \{g \in R^m | ag \in N\}$.

Theorem 5: [125] Let $F \in R^{\ell \times m}$, $d = \mathrm{d}(F)$. If F admits a ZLP factorization $F = G_1 F_1$ with $G_1 \in R^{\ell \times \ell}$, $\det G_1 = d$ and $F_1 \in R^{\ell \times m}$ being ZLP, then the following is true:

1) $\rho(F_1) = \rho(F) : d$ is a free R-module of rank ℓ;
2) $\mathrm{Torsion}(R^m/\rho(F)) = \rho(F_1)/\rho(F)$;
3) $R^m/\rho(F_1)$ is a free R-module, where $\rho(F)$ denotes the submodule generated by all of the row vectors of F.

As for the existence of MLP factorization for n-D polynomial matrices, we recall the following recent result [126].

Theorem 6: [126] Let $F \in R^{\ell \times m}$ and $d = \mathrm{d}(F)$. Then, the following conditions are equivalent.

1) $\rho(F) : d$ is a free module of rank ℓ.
2) there exist polynomial matrices $F_1 \in R^{\ell \times m}$ and $G_1 \in R^{\ell \times \ell}$ such that $F = G_1 F_1$ with $\det G_1 = d$ and F_1 being MLP.

Moreover, some interesting properties concerning MLP factorizations given in [126] are recalled here.

Theorem 7: [126] Let $F \in R^{\ell \times m}$, $d = \mathrm{d}(F)$. If F admits an MLP factorization $F = G_1 F_1$ with $G_1 \in R^{\ell \times \ell}$, $\det G_1 = d$, and $F_1 \in R^{\ell \times m}$ being MLP, then the following is true:

1) $\rho(F_1) = \rho(F) : d$ is a free R-module of rank ℓ;
2) $\mathrm{Torsion}(R^m/\rho(F_1)) = \rho(F_1)/\rho(F)$;
3) $R^m/\rho(F_1)$ is a torsion-free R-module.

Remark 1: Note the similarity and difference between ZLP and MLP factorizations. For both ZLP and MLP factorizations, we have that $\rho(F) : d$ is a free R-module of rank ℓ. For ZLP factorizations, $R^m/\rho(F_1)$ is a free R-module. However, for MLP factorizations, $R^m/\rho(F_1)$ is only a torsion-free R-module, but not a free R-module.

Remark 2: Another difference between ZLP and MLP factorizability is that the existence of ZLP factorization is completely characterized by the reduced minors of the matrix under consideration while there does not appear to have a similar characterization of MLP factorizability in terms of the reduced minors. A sufficient (but not necessary) condition for the MLP factorizability is that the reduced minors $b_1, \cdots, b_\beta$, and d together generate the unit ideal R.

To carry out the actual ZLP and MLP factorizations by Theorems 6 and 7, respectively, one would need to obtain the R-module $\rho(F) : d$. Given a general full-row-rank matrix $F \in R^{\ell \times m}$ and $d = \mathrm{d}(F)$, it seems that there is currently no algorithm available for constructing the required R-module $\rho(F) : d$. For this reason, Theorems 6 and 7 may only be considered as a proof for the existence of ZLP and MLP factorizations, rather than as constructive methods for carrying out the actual factorizations. Constructive algorithms for carrying out the ZLP and MLP factorizations for special classes of n-D polynomial matrices were presented in [10], [24], [63], [67], and [69]. A more general factorization algorithm applicable to both ZLP and MLP factorizations due to Wang and Kwong [126] is briefly described in the following.

Algorithm 1: [126] Let $F \in R^{\ell \times m}$ with $d = \mathrm{d}(F)$. Complete the following algorithm.

Step 1) Use Gröbner bases to compute a system of generators for $\mathrm{Syz}([F^T - dI_m]^T)$.
Step 2) If the number of elements in the system of generators cannot be reduced to ℓ, exit.
Step 3) Partition the $\ell \times (\ell + m)$ matrix that generates the syzygy module as $[G_1\ F_1]$, where $G_1 \in R^{\ell \times \ell}$ and $F_1 \in R^{\ell \times m}$.
Step 4) Make $G_0 F_1$ the required ZLP or MLP factorization for F, where $G_0 = (dG_1^{-1}) \in R^{\ell \times \ell}$.

Note that, in step 4) of the above algorithm, F_1 is ZLP if the reduced minors are zero coprime. Otherwise, F_1 is only MLP. It is also important to point out that presently there is no algorithm that guarantees the construction of ℓ elements for generating the above syzygy module, even by the Gröbner bases. Hence, the problem of ZLP and MLP factorizations for n-D ($n > 2$) polynomial matrices has not yet been fully resolved, at least from the point of view of constructive computations.

The following illustrative example is adopted and modified from [126].

Example 6: Let

$$F = \begin{bmatrix} z_1 z_2^2 z_3 & 0 & -z_1^2 z_2^2 - 1 \\ z_1^2 z_3^2 + z_3 & -z_3 & -z_1^3 z_3 - z_1 \end{bmatrix} \in \mathbf{C}[z_1, z_2, z_3]^{2\times 3}.$$

All of the 2×2-minors are computed as follows:

$$\begin{aligned} a_1 &= -z_1 z_2^2 z_3^2 \\ a_2 &= -z_1^2 z_2^2 z_3 - z_3 \\ a_3 &= z_1^2 z_3^2 + z_3. \end{aligned}$$

Thus, $d = \mathrm{d}(F) = z_3$, and the reduced minors have no common zeros in $\mathbf{C}^3$.

Let $\boldsymbol{s}_i$ be the ith row of $[F^T - z_3 I_3]^T$, $i = 1, \dots, 5$. We list $\boldsymbol{s}_i$ as follows:

$$\begin{aligned}\boldsymbol{s}_1 &= [z_1 z_2^2 z_3, 0, -z_1^2 z_2^2 - 1]\\ \boldsymbol{s}_2 &= [z_1^2 z_3^2 + z_3, -z_3, -z_1^3 z_3 - z_1]\\ \boldsymbol{s}_3 &= [-z_3, 0, 0]\\ \boldsymbol{s}_4 &= [0, -z_3, 0]\\ \boldsymbol{s}_5 &= [0, 0, -z_3].\end{aligned}$$

Now we compute a syzygy module of $\boldsymbol{s}_1, \cdots, \boldsymbol{s}_5$ under the default module term ordering (DegRevlex and ToPos) using CoCoA [30]. It has the following three generators:

$$\begin{aligned}\boldsymbol{f}_1 &= [0, z_3, z_1^2 z_3^2 + z_3, -z_3, -z_1^3 z_3 - z_1]\\ \boldsymbol{f}_2 &= [z_3, -z_1 z_2^2 z_3, -z_1^3 z_2^2 z_3^2, z_1 z_2^2 z_3, z_1^4 z_2^2 z_3 - 1]\\ \boldsymbol{f}_3 &= [z_1, -z_1^2 z_2^2 - 1, -z_1^4 z_2^2 z_3 - z_1^2 z_3 - 1, z_1^2 z_2^2 + 1,\\ &\quad z_1^5 z_2^2 + z_1^3].\end{aligned}$$

Let M be the module generated by $\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3$. Using a CoCoA command "Minimalized(M)," we find that M has only two generators $\boldsymbol{f}_2$ and $\boldsymbol{f}_3$ given above (i.e., $\boldsymbol{f}_1$ is removed). Hence, by Theorem 7, we can obtain a factorization of F as follows.

Let $G' = \begin{bmatrix} z_3 & -z_1 z_2^2 z_3 \\ z_1 & -z_1^2 z_2^2 - 1 \end{bmatrix}$. Because of $\det G' = -z_3$, we set $\boldsymbol{f}_3' = -\boldsymbol{f}_3$, and thus we have that $\boldsymbol{f}_2, \boldsymbol{f}_3'$ is a globally minimal generating system.

Let $G_1 = \begin{bmatrix} z_3 & -z_1 z_2^2 z_3 \\ -z_1 & z_1^2 z_2^2 + 1 \end{bmatrix}$. Thus, we have that $G_0 = dG_1^{-1} = \begin{bmatrix} z_1^2 z_2^2 + 1 & z_1 z_2^2 z_3 \\ z_1 & z_3 \end{bmatrix}$, with $\det G_0 = z_3$.

Let
$F_1 = \begin{bmatrix} -z_1^3 z_2^2 z_3^2 & z_1 z_2^2 z_3 & z_1^4 z_2^2 z_3 - 1 \\ z_1^4 z_2^2 z_3 + z_1^2 z_3 + 1 & -z_1^2 z_2^2 - 1 & -z_1^5 z_2^2 - z_1^3 \end{bmatrix}$, and then $F = G_0 F_1$.

Finally, since the reduced minors of F are zero coprime, the above factorization $F = G_0 F_1$ is a ZLP factorization.

It should be pointed out that very recently Quadrat and coworkers have made considerable efforts in developing interesting constructive methods for the ZLP factorization problem for n-D polynomial matrices and related problems in n-D control systems using the sophisticated tool of module theory and Gröbner bases [35], [36]. We refer the reader to [35] and [36] for more details.

The next important topic is the FLP factorization problem. This is a harder and more challenging open problem mentioned in [9], [54], and [140]. This problem is in fact related to the ZLP and MLP factorization problems discussed earlier and are closely related to the so-called general n-D polynomial matrix factorization problem stated below [67], [127].

Definition 13: [67], [127] Let $F \in R^{\ell \times m}$, and f be a divisor of $\mathrm{d}(F)$, that is, $a_i = f b_i$, where $a_1, \dots, a_\beta$ are all of the $\ell \times \ell$-minors of F, and $b_i \in R$, $i = 1, \dots, \beta$ (not necessarily factor coprime). We say that F admits a matrix factorization w.r.t. f if F can be factorized as

$$F = G_1 F_1 \tag{34}$$

such that $F_1 \in R^{\ell \times m}$, $G_1 \in R^{\ell \times \ell}$, with $\det G_1 = f$.

To the best of our knowledge, Lin *et al.* [67] were the first to attack this general n-D polynomial matrix factorization problem and to obtain some interesting results on the factorization of several special classes of n-D polynomial matrices. Motivated by the work of Lin *et al.* [67] and based on his recent work in [125] and [126], Wang recently proposed another scheme for solving the above n-D polynomial matrix factorization problem for a bigger class of n-D polynomial matrices [127]. Furthermore, Wang's method is also applicable to the FLP factorization problem and hence is described briefly in the following. Note that Gröbner bases and the Quillen-Suslin theorem were exploited in both [67] and [127].

We first recall a necessary definition [127].

Definition 14: [127] Let $F \in R^{\ell \times m}$, $d = \mathrm{d}(F)$, and $f \in R$. f is said to be regular w.r.t. F iff $\mathrm{d}([f I_\ell \; F]) = f$ up to multiplication by a nonzero constant.

Since $d = \mathrm{d}(F)$, for a given f to be regular w.r.t. F, f can only be a divisor of d. To test whether f is regular, we can easily apply Gröbner bases to extract the g.c.d. of the maximal minors of the matrix $[f I_\ell \; F]$.

Theorem 8: [127] Let $F \in R^{\ell \times m}$, $d = \mathrm{d}(F)$, and f be regular w.r.t. F. The following are equivalent.

1) F can be factorized as $F = G_1 F_1$ such that $G_1 \in R^{\ell \times \ell}$, $F_1 \in R^{\ell \times m}$ with $\det G_1 = f$.
2) $\rho(F) : f$ is a free R-module of rank ℓ.

The above theorem gives a necessary and sufficient condition for the existence of general n-D polynomial matrix factorizations for a class of n-D polynomial matrices, i.e., in the case where f is regular w.r.t. F.

To address the FLP factorization problem, we need the following notation from [127]. Let $h, g \in R$, if there is a $r \in R$ with $\deg r \geq 1$ such that $h = gr$, and then h is called a proper multiple of g. For $g \in R$, $F \in R^{\ell \times m}$, and $d = \mathrm{d}(F)$, set $M(g) = \{\forall f \in R \text{ such that } g | f \text{ and } f | \mathrm{d}(F)\}$.

The following important theorem of [127] is recalled here.

Theorem 9: [127] Let $F \in R^{\ell \times m}$ and $g \in R$ be a divisor of $\mathrm{d}(F)$. Assume that, for each $f \in M(g)$, f is regular w.r.t. F. Then the following conditions are equivalent.

1) F has an FLP factorization w.r.t. g.
2) $\rho(F) : g$ is free, and $\rho(F) : h$ is not free for any $h \in M(g)$ which is a proper multiple of g.

Theorem 9 gives a partial solution to the FLP factorization problem. Note that this theorem turns out to be quite similar to Theorem 6. Hence, Algorithm 1 can also be used for the computation of FLP factorization with the replacement of d in Algorithm 1 with g here. The details are hence omitted.

IV. Applications of Gröbner Bases in Other Areas of Multidimensional Signals and Systems

In the previous section, we gave a reasonably detailed review of several fundamental theoretical issues that occur in the use of Gröbner bases in multidimensional signals and systems. This section considers several other important areas of multidimensional signals and systems. Due to the wide range of applications of Gröbner bases in these areas and the limited space available here, the review has to be brief, but we do hope that the impact and scope of Gröbner bases in these areas will motivate further research.

1) n-D Control System Analysis, Synthesis, and Realization: As far as we are aware, the earliest application of Gröbner bases to n-D systems is in the area of feedback stabilizability and stabilization of 2-D systems by Guiver and Bose [46], followed by other results on the subject in the later half of the 1980s [7], [38]. The main reason why Gröbner bases could be successfully applied in feedback stabilizability and stabilization of 2-D systems is that the problem can be reduced to the finding of a solution to a linear Diophantine equation in the ring of 2-D polynomials. Research activities in this direction continued into the 1990s with new methods developed for directly solving the strong stabilizability of n-D systems by Ying *et al.* [139], for the problem of stabilizability and stabilization of multi-input multi-output (MIMO) 2-D systems using Gröbner bases of modules by Xu *et al.* [135], and for special classes of MIMO n-D $(n > 2)$ systems using Gröbner bases [66], [81], [134]. Very recently, Quadrat has solved the open problem of stabilizability and stabilization for a general MIMO n-D $(n > 2)$ system using a lattice approach [103].

For n-D systems representation and realization, Gröbner bases have been applied as a tool for obtaining canonical state-space representation and absolutely minimal state-space realization [87], [136].

For n-D nonlinear control systems, Gröbner bases have also found good applications due to its power in solving polynomial systems in several variables like the one given in (1). See [13], [40], [82], [83], [100], and for more details.

For other n-D control system analysis and synthesis problems where Gröbner bases have played an important role such as tracking control, disturbance rejection, and observer design, see [137] and and the references therein.

2) Behavioral and Module Theory for n-D Systems: In his seminal paper [85], Oberst demonstrated how some fundamental problems of behavioral n-D systems theory can be reduced to the construction of Gröbner bases. In particular, a duality theorem has been established which links n-D linear systems with finitely generated modules over n-D polynomial algebra [85]. Consequently, "Gröbner bases are invaluable for almost any non-trivial constructive exercise (in behavioral n-D system theory)", as remarked by Pillai *et al.* in [101] (see also [132]). In the last decade or so, there are considerable research efforts on this area, and some of the related papers are listed here [41], [86], [87], [131], [145], [146].

3) n-D Wavelets, Filter Banks, and Filters: It is no doubt that the analysis and design of n-D wavelets, filter banks, and filters is a very important area in n-D signal processing and communications. An effective way to describe an n-D filter bank is to represent it with an n-D (Laurent) polynomial matrix for the FIR case and with an n-D rational matrix for the IIR case. Because of its obvious advantage in the manipulation of n-D polynomial and rational matrices, the theory of Gröbner bases has found increasing applications in n-D wavelets, filter banks, and filters since the early 1990s; see, e.g., [3], [4], [37], and [94]–[96]. For a survey in this area up to 2004, see [68] and [98]. Continued interests in recent years include the optimal construction of compactly supported n-D wavelets [99] and the 2-D antisymmetric linear phase filter bank construction using symmetric completion [147]. In n-D digital filters, it has been shown that the stability test and stability margin computation problems can be formulated in a unified way as a system of algebraic equations characterized by $n+1$ polynomials in $n+1$ variables, which can be solved using the Gröbner bases approach [25], [32].

4) n-D Coding and Deconvolution: Applications of Gröbner bases to n-D coding has a long history dating back to early 1990 (see, for example, [111] and [112]). As n-D coding involves n-D polynomial matrices, Gröbner bases theory is a powerful tool for this challenging area (see [128] for a detailed discussion). Recent work in this area can be found in [26] and [27], and we hope to see more work along this direction in the near future. n-D multichannel deconvolution has many applications such as channel equalization for multiple antennas, multichannel image deconvolution, and polarimetric calibration of radars. Traditional method for the n-D multichannel FIR deconvolution is based on numerical linear algebra. Recently, Gröbner bases have been successfully applied to solving this problem, and a complete characterization of all exact deconvolution FIR filters is available [148]. Note that the authors of [148] are apparently unaware of the Rabinowitsch trick, used before in n-D systems theory [135] (see, for example, [148, proof of Theorem 2]). This shows the usefulness and importance of putting together here major papers applying Gröbner bases to different areas.

V. Applications of Gröbner Bases in 1-D Signals and Systems

The applications of Gröbner bases are not restricted to n-D signals and systems. In fact, there are many applications of Gröbner bases in diverse areas in 1-D signals and systems. In this section, we give a brief review of these applications and refer the reader to the cited references for more details.

1) Control and Systems: Gröbner bases have been successfully applied in various areas of control and systems in the past decades. One application is the analytic characterization and optimal minimax control solution using rate feedback reported in [11] and more recently in [28]. Another application is the computation of switching surfaces in optimal control [124]. Finding an optimal solution to the L_2 control system approximation problem is yet another application [47]. In [48], Helton *et al.* considered the rather general issue of computer simplification of formulas in linear systems theory, using a noncommutative version of the Gröbner basis algorithm. The topics covered in [48] were quite wide, including H^{infty} control and Lyapunov equations. More recent applications of Gröbner bases include the model-based fault detection in a centrifugal pump control system [53].

2) Circuits and Networks: Applications of Gröbner bases can be found in diverse areas in circuits and networks, such as the solution to the load-flow problem as the changes connected with an electrical network vary [78], the algebraic representation of error bounds for describing function applying to nonlinear circuits and systems [138], and the development of efficient symbolic procedures for testability measures and ambiguity groups determination for analog linear circuits [22].

3) Wavelets, Filter Banks, and Filters: In parallel to the applications of Gröbner bases to the analysis and design of n-D wavelets, filter banks, and filters, Gröbner bases have also found extensive applications in the 1-D counterparts. For research work on this topic up to 2004, see [59], [61], [68], [70], [75], and [113] and the references therein. In a recent paper, Regensburger and Scherzer [105] established explicit and not recursive, as done earlier, bijective relations between continuous and discrete moments of scaling functions associated with orthogonal wavelets by expressing the nth continuous moment as a polynomial (related to a class of distinguished polynomials, called Bell polynomials) of the first n discrete moments and *vice versa*. Subsequently, they parameterized the filter coefficients in the dilation equations generating the scaling equation and compactly supported orthonormal wavelets with several vanishing moments, in terms of the discrete (and, consequently, the continuous) moments. By giving up certain vanishing moments to obtain additional degrees of freedom, they were able to generate a parameterized solution set in contrast to previous methods on symbolic computation of wavelets coefficients (which calculated a finite number of solutions) and used algorithmic algebraic tools like Gröbner bases. The results in [105] are significant because previous approaches to parameterization expressed the filter coefficients in terms of trigonometric functions (and not polynomials) which required the solving of transcendental constraints for the parameters to find wavelets with more than one vanishing moment.

4) Signal Reconstruction and Restoration: Very recently, Gröbner bases have been applied to signal reconstruction from multiple sets of samples with unknown offsets [121]. The exact inversion of MIMO nonlinear polynomial mixtures that arise in signal restoration is considered in [23].

5) Coding and Decoding: The search for theoretically better codes, for which efficient decoding algorithms can be found, leads to further consideration of many other codes like quasicyclic codes and AG codes (where data of very high reliability is desired by the user, like optical communication links). In algebraic coding theory, Gröbner bases have already been quite extensively used to satisfy the objective stated. Unlike in other applications considered, the field of coefficients in coding is a finite field. The history of applying Gröbner bases to coding and decoding is a long one, and there are many papers published in this area. Here we only cite a few recent journal papers and refer the reader to these papers for more references. In the area of coding, see, e.g., [43], [80], [84], [88], and [89]. In the area of decoding, see, e.g., [20], [21], [29], [60], and [76].

6) Robotics: Early application of Gröbner bases in robotics dated back to the 1990s. For example, Gröbner bases were applied to solve kinematic equations arising in the analysis and design of robots and other linkage systems [52]. More recent applications of Gröbner bases in robotics include the exact solutions to the forward kinematics problem specially applied to spatial and planar parallel manipulator in robotics [109], [110].

7) Other Applications: Gröbner bases have also been applied to many other areas, including, for example, the annihilation of loading parameters in classical numerical methods which find various applications in applied mechanics and engineering problems [51], the analysis of flexible link systems [120], the design of railway interlocking system to prevent conflicting actions [106], [108], and the checking of multivalued models which is very useful in reasoning about the correctness of hardware, communication protocols, and software requirements [133].

VI. Discussions and Conclusion

The resultant-based approach to solving a variety of problems in multidimensional signals and systems is extensively documented [9]. However, it is beginning to be felt that Gröbner bases comprise the most powerful computational technique in commutative algebra and its applications in circuits, control, coding, signals, and systems problems. It has also been acknowledged that the worst-case complexity for computing Gröbner bases for polynomials in even three or four indeterminates is high, but the expected (average) complexity, especially in structured problems, may be quite acceptable. Less computation-intensive algorithms for computing Gröbner bases will stimulate their applications to moderate and large-sized, especially parameterized, problems in a large number of indeterminates and parameters. For such problems, a recourse now is to reduce the number of variables by computing the resultant of pairs of polynomials to eliminate, at each step, a variable.

A vast arena for application of Gröbner bases concerns nonconvex optimization problems. For a survey of solving such problems that can be formulated as global optimization problems with polynomial objective functions, occurring in robust and nonlinear control, by classical and more modern methods, see [49]. The potential for using the resources in the theory of Gröbner bases has recently been pointed out in [13], [100], and . However, just the surface has been scratched, and further applications in diverse areas with concurrent development of more efficient theoretical and computational tools are on the horizon. Since the property of multivariate polynomial positivity (implied by the sum-of-squares (SOS) representation but not vice versa) is at the heart of fundamental problems like stability, robustness, and performance, there is a genuine need to supplement the resources provided by tools like SOSTOOLS (Sum of Squares Optimization Toolbox for MATLAB), LMI [14], and SeDuMi (for optimization over symmetric cones) by those available in the algorithmic algebra of Gröbner bases. Not only can one visualize expanding domains of applications, but also simultaneous attention to theoretical development as well as numerical issues of implementation. At present, the applications of Gröbner bases appear to be restricted to problems of small to moderate size, and this has to be improved upon in the future for a wider acceptance.

Challenging problems of a more mathematical nature, the solution of which would impact several applications, remain to be tackled. For example, the open problem of factor prime factorizations for n-D polynomial matrices has not been completely resolved. Another example is the solution to the unimodular completion problem when constraints are placed. For example, multidimensional lossless filter banks are known to be characterized by paraunitary matrices with multivariate polynomial entries. However, the constructive solutions available for the unimodular completion problem have not led to any complete solution when the constraint of paraunitariness is imposed [95]. Only some partial solutions appear to be available [99].

Other than those areas reviewed so far in which signals and systems engineers and researchers have no doubt played an important role, it is worth pointing out that Gröbner bases have also found important applications in emerging areas such as computational biology, in which signals and systems engineers and researchers are beginning to show interested. For example, in the recent book [91] on algebraic statistics for computational biology, Gröbner bases were recommended as one of the major tools in algebra for solving problems in computational biology, such as in mutagenetic tree models for accumulative evolutionary processes. The reader is referred to [91] for a more detailed discussion of how Gröbner bases can play a major role in this emerging and truly interdisciplinary area.

Finally, despite all of the advantages and potentials of Gröbner bases, the reader should also be reminded of some limitations of Gröbner bases. In some worst cases, the number of elements in a Gröbner basis of a given ideal can be significantly larger than that of another basis of the same ideal, making a Gröbner basis not very useful in applications where a smaller number of elements is preferred in a basis. Another limitation is that current software packages implementing Gröbner bases still cannot deal with very large problems with symbolic coefficients, although tremendous progress has been made recently in this direction.

Acknowledgment

The authors would like to thank the Editor-in-Chief, Dr. S. Basu, for his invitation to write this paper and for many helpful discussions. Gratitude is expressed to Prof. B. Buchburger, Dr. M. Wang, and the other three anonymous reviewers for their careful reading of this paper and many constructive comments. The writing was stimulated by the Special Semester on Gröbner Bases supported by RICAM (the Radon Institute for Computational and Applied Mathematics, Austrian Academy of Science, Linz) and organized by RICAM and RISC (Research Institute for Symbolic Computation, Johannes Kepler University, Linz, Austria) under the scientific direction of Prof. B. Buchberger.

References

[1] W. W. Adams and P. Loustaunau, *An Introduction to Gröbner Bases*. Providence, RI: Amer. Math. Soc., 1994, vol. 3, Graduate Studies in Mathematics.

[2] H. M. Antia, *Numerical Methods for Scientists and Engineers*. New York: Birkhauser, 2002.

[3] S. Basu, "Multidimensional filter banks and wavelets—A system theoretic perspective," *J. Franklin Inst.*, vol. 335B, no. 8, pp. 1367–1409, 1998.

[4] S. Basu, "Multidimensional causal, stable, perfect reconstruction filter banks," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 49, no. 6, pp. 832–842, Jun. 2002.

[5] S. Basu and M. N. S. Swamy, Eds., *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 49, Special Issue on Multidimensional Signals and Systems, no. 6, pp. 709–891, Jun. 2002.

[6] T. Becker and V. Weisfenning, *Gröbner Bases*. Berlin, Germany: Springer-Verlag, 1993.

[7] M. Bisiacco, E. Fornasini, and G. Marchesini, "Causal 2-D compensators: Stabilization algorithms for multivariable 2-D systems," in *Proc. 25th Conf. Decision Control*, 1986, pp. 2171–2174.

[8] R. B. Bhat and S. Chakraverty, *Numerical Analysis in Engineering*. Pangbourne, U.K.: Alpha Sci., 2004.

[9] N. K. Bose, *Applied Multidimensional Systems Theory*. New York: Van Nostrand Reinhold, 1982.

[10] N. K. Bose and C. Charoenlarpnopparut, "Multivariate matrix factorization: New results," in *Proc. MTNS*, Padova, Italy, Jul. 1998, pp. 97–100.

[11] N. K. Bose and C. Charoenlarpnopparut, "Minimax controller design using rate feedback," *Circuits, Syst. Signal Process.*, vol. 18, no. 1, pp. 17–25, 1999.

[12] N. K. Bose, B. Buchberger, and J. P. Guiver, *Multidimensional Systems Theory and Applications*. Dordrecht, The Netherlands: Kluwer, 2003.

[13] N. K. Bose, "Two decades (1985–2005) of Gröbner bases in multidimensional systems," in *Proc. Workshop on GB in Contr. Theory Signal Process.*, Linz, Austria, May 2006.

[14] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge Univ. Press, 2004.

[15] B. Buchberger, "An algorithm for finding the basis elements of the residue class ring modulo a zero-dimensional polynomial ideal (German)," Ph.D. dissertation, Dept. of Math., Univ. of Innsbruck, Innsbruck, Austria, 1965.

[16] B. Buchberger, "An algorithmic criterion for the solvability of algebraic systems of equations (German)," *Aequationes Mathematicae*, vol. 4, no. 3, pp. 374–383, 1970, (English translation in [18]).

[17] B. Buchberger, "Gröbner bases: An algorithmic method in polynomial ideal theory," in *Multidimensional Systems Theory: Progress, Directions and Open Problems*, N. K. Bose, Ed. Dordrecht, The Netherlands: Reidel, 1985, pp. 184–232.

[18] B. Buchberger and F. Winkler, Eds., *Gröbbner Bases and Applications*, ser. London Mathematical Society Series 457. Cambridge, U.K.: Cambridge Univ. Press, 1998, no. 500.

[19] B. Buchberger, "Gröbner bases and systems theory," *Multidimensional Syst. Signal Process.*, vol. 12, pp. 223–251, 2001.

[20] E. Byrne and P. Fitzpatrick, "Hamming metric decoding of alternant codes over Galois rings," *IEEE Trans. Inf. Theory*, vol. 48, no. 3, pp. 683–694, Mar. 2002.

[21] E. Byrne, "Decoding a class of Lee metric codes over a Galois ring," *IEEE Trans. Inf. Theory*, vol. 48, no. 4, pp. 966–975, Apr. 2002.

[22] B. Cannas, A. Fanni, and A. Montisci, "Testability evaluation for analog linear circuits via transfer function analysis," in *Proc. ISCAS*, Kobe, Japan, May 2005, pp. 992–995.

[23] M. Castella, "Exact inversion of MIMO nonlinear polynomial mixtures," in *Proc. ICASSP*, Honolulu, HI, 2007, vol. III, pp. 1429–1432.

[24] C. Charoenlarpnopparut and N. K. Bose, "Multidimensional FIR filter bank design using Gröbner bases," *IEEE Trans. Circuits Syst. II, Analog Digit. Signal Process.*, vol. 46, no. 11, pp. 1475–1486, Nov. 1999.

[25] C. Charoenlarpnopparut and N. K. Bose, "Gröbner bases for problem solving in multidimensional systems," *Multidimensional Syst. Signal Process.*, vol. 12, pp. 365–376, 2001.

[26] C. Charoenlarpnopparut and S. Tantaratana, "Multidimensional convolutional code: Progresses and bottlenecks," in *Proc. ISCAS*, Bangkok, Thailand, May 2003, vol. III, pp. 686–689.

[27] C. Charoenlarpnopparut and S. Tantaratana, "Algebraic approach to reduce the number of delay elements in the realization of multidimensional convolutional code," in *Proc. MWSCAS*, Hiroshima, Japan, Jul. 2004, pp. II-529–II-532.

[28] C. Charoenlarpnopparut, "Optimal minimax control solution using rate feedback," in *Proc. ICICS*, Bangkok, Thailand, Dec. 2005, pp. 6–9.

[29] J.-P. Chen and C.-C. Lu, "A serial-in-serial-out hardware architecture for systematic encoding of Hermitian codes via Grobner bases," *IEEE Trans. Commun.*, vol. 52, no. 8, pp. 1322–1332, Aug. 2004.

[30] CoCoA Team, CoCoA, a System for Doing Computations in Commuatative Algebra. [Online]. Available: http://www.cocoa.dima.unige.it

[31] D. Cox, J. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms*. Berlin, Germany: Springer-Verlag, 1992.

[32] E. Curtin and S. Saba, "Stability and margin of stability tests for multidimensional filters," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 46, pp. 806–809, 1999.

[33] D. Eisenbud, *Commutative Algebra With a View Towards Algebraic Geometry*. New York: Springer-Verlag, 1995.

[34] E. G. Evans and P. Griffith, *Syzygies*. Cambridge, U.K.: Cambridge Univ. Press, 1985.

[35] A. Fabiańska and A. Quadrat, "Flat multidimensional linear systems with constant coefficients are equivalent to controllable 1-D linear systems," in *Proc. MTNS*, Kyoto, Japan, Jul. 2006, pp. 24–28.

[36] A. Fabiańska and A. Quadrat, Applications of the Quillen-Suslin Theorem to Multidimensional Systems Theory INRIA Rep. 6126, 2007.

[37] J.-C. Faugere, F. M. de Saint-Martin, and F. Rouillier, "Design of regular nonseparable bidimensional wavelets using Gröbner basis techniques," *IEEE Trans. Signal Process.*, vol. 46, no. 6, pp. 845–856, Jun. 1998.

[38] E. Fornasini, "A note on output feedback stabilizability of multivariable 2-D systems," *Syst. Control Lett.*, vol. 10, pp. 45–50, 1988.

[39] E. Fornasini and M. E. Valcher, " nD polynomial matrices with applications to multidimensional signal analysis," *Multidimensional Syst. Signal Process.*, vol. 8, pp. 387–408, Oct. 1997.

[40] I. A. Fotiou, P. Rostalski, P. A. Parrilo, and M. Morari, "Parametric optimization and optimal control using algebraic geometry methods," *Int. J. Control*, vol. 79, no. 11, pp. 1340–1358, Nov. 2006.

[41] S. Fröhler and U. Oberst, "Continuous time-varying linear systems," *Syst. Control Lett.*, vol. 35, no. 2, pp. 97–110, 1998.

[42] A. Furukawa, T. Sasaki, and H. Kobayashi, "Gröbner basis of a module over $K[x_1, \ldots, x_n]$ and polynomial solutions of a system of linear equations," in *Proc. SYMSAC*, B. W. Char, Ed., Waterloo, ON, Canada, 1986, pp. 222–224.

[43] O. Geil, "On codes from norm-trace curves," *Finite Fields Their Appl.*, vol. 9, pp. 351–371, 2003.

[44] K. O. Geddes, S. R. Czapor, and G. Labahn, *Algorithms for Computer Algebra*. Dordrecht, The Netherlands: Kluwer, 1992.

[45] J. P. Guiver and N. K. Bose, "Polynomial matrix primitive factorization over arbitrary coefficient field and related results," *IEEE Trans. Circuits Syst.*, vol. CST-29, no. 3, pp. 649–657, May 1982.

[46] J. P. Guiver and N. K. Bose, "Causal and weakly causal 2-D filters with applications in stabilizations," in *Multidimensional Systems Theory: Progress, Directions and Open Problems*, N. K. Bose, Ed. Dordrecht, The Netherlands: Reidel, 1985, pp. 52–100.

[47] B. Hanzon and J. M. Maciejowski, "Constructive algebra methods for the L2-problem for stable linear systems," *Automatica*, vol. 32, no. 12, pp. 1645–1657, 1996.

[48] J. W. Helton, M. Stankus, and J. J. Wavrik, "Computer simplification of formulas in linear systems theory," *IEEE Trans. Autom. Control*, vol. 43, no. 3, pp. 302–314, Mar. 1998.

[49] D. Henrion and J.-B. Lasserre, "Solving nonconvex optimization problems," *IEEE Control Syst. Mag.*, vol. 24, pp. 72–83, 2004.

[50] H. Hironaka, "Resolution of singularities of an algebraic variety over a field of characteristic zero: I," *Annals Math.*, ser. 2, vol. 79, no. 1, pp. 109–203, Jan. 1964.

[51] N. I. Ioakimidis and E. G. Anastasselou, "Annihilation of loading parameters in classical numerical methods with differential equations," *Comput. Structures*, vol. 59, no. 2, pp. 265–271, 1996.

[52] C. M. Kalker-Kalkman, "An implementation of Buchbergers' algorithm with applications to robotics," *Mechanism Mach. Theory*, vol. 28, no. 4, pp. 523–537, 1993.

[53] C. S. Kallesoe, V. Cocquempot, and R. Izadi-Zamanabadi, "Model based fault detection in a centrifugal pump application," *IEEE Trans. Control Syst. Technol.*, vol. 14, no. 2, pp. 204–215, Mar. 2006.

[54] S. Kleon and U. Oberst, "Transfer operators and state spaces for discrete multidimensional linear systems," *Acta Applicandae Mathematicae*, vol. 57, pp. 1–82, 1999.

[55] M. Kreuzer and L. Robbiano, *Computational Commutative Algebra 1*. Berlin, Germany: Springer-Verlag, 2000.

[56] E. Kreyszig, *Advanced Engineering Mathematics*, 9th ed. New York: Wiley, 2006.

[57] T. Y. Lam, *Serre's Conjecture*. Berlin, Germany: Springer-Verlag, 1978, vol. 635, Lecture Notes in Mathematics.

[58] T. Y. Lam, *Serre's Problem on Projective Modules*. Berlin, Germany: Springer-Verlag, 2006.

[59] J. Lebrun and I. Selesnick, "Gröbner bases and wavelet design," *J. Symbol. Computat.*, vol. 37, no. 2, pp. 227–259, 2004.

[60] K. Lee and M. E. O'Sullivan, "An interpolation algorithm using Gröbner bases for soft-decision decoding of Reed–Solomon codes," in *Proc. IEEE Int. Symp. Inf. Theory*, Jul. 2006, pp. 2032–2036.

[61] F. Leeb and T. Henk, "Simultaneous amplitude and phase approximation for FIR filter design," *Int. J. Circuit Theory Appl.*, vol. 17, pp. 363–374, Jul. 1989.

[62] Z. Lin, "On matrix fraction descriptions of multivariable linear n-D systems," *IEEE Trans. Circuits Syst.*, vol. 35, no. 11, pp. 1317–1322, Nov. 1988.

[63] Z. Lin, "Notes on n-d polynomial matrix factorizations," *Multidimensional Syst. Signal Process.*, vol. 10, pp. 379–393, 1999.

[64] Z. Lin and N. K. Bose, "A generalization of Serre's conjecture and some related issues," *Linear Algebra Appl.*, vol. 338, pp. 125–138, 2001.

[65] Z. Lin and L. Xu, Eds., *Multidimensional Syst. Signal Process.*, vol. 12, Special Issue on Applications of Gröbner Bases to Multidimensional Systems and Signal Processing, pp. 215–381, 2001.

[66] Z. Lin, J. Lam, K. Galkowski, and S. Xu, "A constructive approach to stabilizability and stabilization of a class of nD systems," *Multidimensional Syst. Signal Process.*, vol. 12, pp. 329–343, 2001.

[67] Z. Lin, J. Q. Ying, and L. Xu, "Factorizations for nD polynomial matrices," *Circuits, Syst. Signal Process.*, vol. 20, pp. 601–618, 2001.

[68] Z. Lin, L. Xu, and Q. Wu, "Applications of Gröbner bases to signal and image processing: A survey," *Linear Algebra Appl.*, vol. 391, no. 1, pp. 169–202, 2004.

[69] Z. Lin, L. Xu, and H. Fan, "On minor prime factorization for n-D polynomial matrices," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 52, no. 9, pp. 568–571, Sep., 2005.

[70] J. B. Little, "Solving the Selesnick-Burrus filter design equations using computational algebra and algebraic geometry," *Adv. Appl. Math.*, vol. 31, no. 2, pp. 463–500, 2003.

[71] A. Logar and B. Sturmfels, "Algorithms for the Quillen-Suslin theorem," *J. Algebra*, vol. 145, pp. 231–239, 1992.

[72] H. Lombardia and I. Yengui, "Suslin's algorithms for reduction of unimodular rows," *J. Symbol. Computat.*, vol. 39, pp. 707–717, 2005.

[73] C. Lossen, "Singular: A computer algebra system," *Computi. Sci. Eng.*, vol. 5, no. 4, pp. 45–55, Jul.–August 2003.

[74] D. Grayson and M. Stillman, Macaulay 2: A Software System for Algebraic Geometry and Commutative Algebra [Online]. Available: www.math.uiuc.edu/Macaulay2

[75] M. Maslen and P. Abbott, "Automation of the lifting factorisation of wavelet transforms," *Comput. Phys. Commun.*, vol. 127, no. 2–3, pp. 309–326, 2000.

[76] H. Matsui, S. Sakata, M. Kurihara, and S. Mita, "Systolic array architecture implementing Berlekamp–Massey–Sakata algorithm for decoding codes on a class of algebraic curves," *IEEE Trans. Inf. Theory*, vol. 51, no. 11, pp. 3856–3871, Nov. 2005.

[77] B. Mishra, *Algorithmic Algebra*. New York: Springer-Verlag, 1993.

[78] A. Montes, "Algebraic solution of the load-flow problem for a 4-nodes electrical network," *Math. Comput. Simulation*, vol. 45, no. 1–2, pp. 163–174, 1998.

[79] F. Mora and H. M. Möller, "New constructive methods in classical ideal theory," *J. Algebra*, vol. 100, no. 1, pp. 138–178, 1986.

[80] T. Mora and M. Sala, "On the Gröbner bases of some symmetric systems and their application to coding theory," *J. Symbol. Computat.*, vol. 35, no. 2, pp. 177–194, 2003.

[81] K. Mori, "A new parameterization method for all stabilizing controllers of nD systems without coprime factorizability," in *Proc. IEEE Symp. Circuits Syst.*, Bangkok, Thailand, 2003, vol. III, pp. 678–681.

[82] D. Nesić and I. M. Y. Mareels, "Dead beat controllability of polynomial systems: Symbolic computation approaches," *IEEE Trans. Autom. Control*, vol. 43, no. 2, pp. 162–175, Feb. 1998.

[83] D. Nesić and I. M. Y. Mareels, "Controllability of structured polynomial systems," *IEEE Trans. Autom. Control*, vol. 44, no. 4, pp. 761–764, Apr. 1999.

[84] G. H. Norton and A. Slgean, "Strong Gröbner bases and cyclic codes over a finite-chain ring," *Electron. Notes Discrete Math.*, vol. 6, pp. 240–250, 2001.

[85] U. Oberst, "Multidimensional constant linear systems," *Acta Applicandae Mathematicae*, vol. 20, pp. 1–175, 1990.

[86] U. Oberst and F. Pauer, "The constructive solution of linear systems of partial difference and differential equations with constant coefficients," *Multidimensional Syst. Signal Process.*, vol. 12, pp. 253–308, 2001.

[87] U. Oberst, "Canonical state representations and Hilbert functions of multidimensional systems," *Acta Appl Math*, vol. 94, pp. 83–135, 2006.

[88] H. Ohsugi, D. Ikegami, T. Kitamura, and T. Hibi, "Gröbner bases of certain zero-dimensional ideals arising in coding theory," *Adv. Appl. Math.*, vol. 31, no. 2, pp. 420–432, 2003.

[89] H. O'Keeffe and P. Fitzpatrick, "Gröbner basis solutions of constrained interpolation problems," *Linear Algebra Appl.*, vol. 351–352, pp. 533–551, 2002.

[90] P. V. O'Neil, *Advanced Engineering Mathematics*, 6th ed. Tampa, FL: Thomson, 2007.

[91] L. Pachter and B. Sturmfels, Eds., *Algebraic Statistics for Computational Biology*. Cambridge, U.K.: Cambridge Univ. Press, 2005.

[92] H. Park, "A computational theory of Laurent polynomial rings and multidimensional FIR systems," Ph.D. dissertation, Dept. Elect. Eng. Comp. Sci., Univ. California, Berkeley, 1995.

[93] H. Park and C. Woodburn, "An algorithmic proof of Suslin's stability theorem for polynomial rings," *J. Algebra*, vol. 178, pp. 277–298, 1995.

[94] H. Park, T. Kalker, and M. Vertterli, "Gröbner bases and multidimensional FIR multirate systems," *Multidimensional Syst. Signal Process.*, vol. 8, pp. 11–30, 1997.

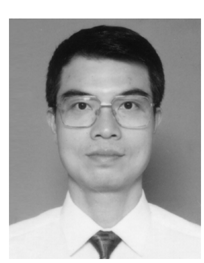
[95] H. Park, "Parametrized family of 2-D non-factorable FIR lossless systems and Gröbner bases," *Multidimensional Syst. Signal Process.*, vol. 12, pp. 345–364, 2001.

[96] H. Park, "Optimal design of synthesis filters in multidimensional perfect reconstruction FIR filter banks using Gröbner bases," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 49, no. 5, pp. 843–851, May 2002.

[97] H. Park, "Prime factorization of n-D polynomial matrices," in *Proc. IEEE ISCAS*, Bangkok, Thailand, May 2003, vol. III, pp. 666–669.

[98] H. Park, "Symbolic computation and signal processing," *J. Symbol. Computat.*, vol. 37, no. 2, pp. 209–226, 2004.

[99] H. Park, "Optimal construction of compactly-supported multidimensional wavelets," in *Proc. IEEE ISCAS*, Kobe, Japan, May 2005, pp. 2417–2420.
[100] P. Parrilo, "Exploiting algebraic structure in sum of squares programs," in *Positive Polynomials in Control*, D. Henrion and A. Garulli, Eds. Berlin, Germany: Springer-Verlag, 2005, pp. 181–194.
[101] H. Pillai, J. Wood, and E. Rogers, "Gröbner bases and constructive multidimensional behavioural theory," in *Proc. 2nd Int. Workshop Multidimensional (nD) Syst.*, Czocha Castle, Poland, Jun. 2000, pp. 83–89.
[102] J. F. Pommaret, "Solving Bose conjecture on linear multidimensional systems," in *Proc. Eur. Control Conf.*, 2001, pp. 1853–1855.
[103] A. Quadrat, "A lattice approach to analysis and synthesis problems," *Math. Control, Signals Syst.*, vol. 18, no. 2, pp. 147–186, May 2006.
[104] D. Quillen, "Projective modules over polynomial rings," *Invent. Math*, vol. 36, pp. 167–171, 1976.
[105] G. Regensburger and O. Scherzer, "Symbolic Computation for Moments and Filter Coefficients of Scaling Functions," *Annals Combinator.*, vol. 9, pp. 223–243, 2005.
[106] E. Roanes-Lozano, E. Roanes-Macías, and L. M. Laita, "Railway interlocking systems and Gröbner bases," *Math. Comput. Simulation*, vol. 51, no. 5, pp. 473–481, 2000.
[107] E. Roanes-Lozano, E. Roanes-Macías, and L. M. Laita, "The geometry of algebraic systems and their exact solving using Gröbner bases," *Computi. Sci. Eng.*, vol. 6, no. 2, pp. 76–79, Mar.–Apr. 2004.
[108] E. Roanes-Lozano, E. Roanes-Macías, and L. M. Laita, "Some applications of Gröbner bases," *Comput. Sci. Eng.*, vol. 6, no. 3, pp. 56–60, May–Jun. 2004.
[109] L. Rolland, "Certified solving of the forward kinematics problem with an exact algebraic method for the general parallel manipulator," *J. Adv. Robot.*, vol. 19, no. 9, pp. 995–1025, Oct. 2005.
[110] L. Rolland, "Synthesis on forward kinematics problem algebraic modeling for the planar parallel manipulator: Displacement-based equation systems," *J. Adv. Robot.*, vol. 20, no. 9, pp. 1035–1065, Sep. 2006.
[111] S. Sakata, "Extension of the Berlekamp-Massey algorithm to N dimensions," *Inf. Computat.*, vol. 84, no. 2, pp. 207–239, 1990.
[112] K. Saints and C. Heegard, "Algebraic-geometric codes and multidimensional cyclic codes: A unified theory and algorithms for decoding using Grobner bases," *IEEE Trans. Inf. Theory*, vol. 41, no. 6, pt. 1, pp. 1733–1751, Nov. 1995.
[113] I. W. Selesnick and C. S. Burrus, "Maximally flat low-pass FIR filters with reduced delay," *IEEE Trans. Circuits Syst. II, Analog Digit. Signal Process.*, vol. 45, no. 1, pp. 53–68, Jan. 1998.
[114] B. Sturmfels, "What is a Groebner Basis?," *Notices of the AMS*, Nov. 2005.
[115] V. R. Sule, "Feedback stabilization over commutative rings: The matrix case," *SIAM J. Control Optim.*, vol. 32, pp. 1675–1695, 1994.
[116] G. M. Greuel, G. Pfister, and H. Schonemann, Singular Reference Manual Reports on Computer Algebra, no. 12, Centre of Algebra, University of Kaiserslautern, Germany [Online]. Available: http://www.mathematik.uni-kl.de/~zca/Singular
[117] V. Srinivas, "A generalized Serre problem," *J. Algebra*, vol. 278, no. 2, pp. 621–627, Aug. 2004.
[118] A. A. Suslin, "Projective modules over a polynomial ring are free," *Sov. Math Dokl.*, vol. 17, pp. 1160–1164, 1976.
[119] W. Trinks, "On B. Buchberger's method for solving systems of algebraic equations," *J. Number Theory*, vol. 10, pp. 475–488, 1978.
[120] F. Ubertini, "A contribution to the analysis of flexible link systems," *Int. J. Solids Structures*, vol. 37, no. 7, pp. 969–990, 2000.
[121] P. Vandewalle, L. Sbaiz, and M. Vetterli, "Signal reconstruction from multiple unregistered sets of samples using Groebner bases," in *Proc. ICASSP*, Tuluse, France, 2006, vol. III, pp. 604–607.
[122] J. von zur Gathen and J. Gerhard, *Modern Computer Algebra*. Cambridge, U.K.: Cambridge Univ. Press, 2003.
[123] B. Wall, "Computation of syzygies, solution of linear systems over a multivariate polynomial ring," Diploma thesis, Res. Inst. for Symbolic Comput., Johannes Kepler Univ., Linz, Austria, 1988.
[124] U. Walther, T. T. Georgiou, and A. Tannenbaum, "On the computation of switching surfaces in optimal control: A Gröbner basis approach," *IEEE Trans. Autom. Control*, vol. 46, no. 4, pp. 534–540, Apr., 2001.
[125] M. Wang and D. Feng, "On Lin–Bose problem," *Linear Algebra Appl.*, vol. 390, pp. 279–285, Oct. 2004.
[126] M. Wang and C. P. Kwong, "On multivariate polynomial matrix factorization problems," *Math. Control Signals Syst.*, vol. 17, no. 4, pp. 297–311, 2005.
[127] M. Wang, "On factor prime factorizations for n-D polynomial matries," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 54, no. 6, pp. 1398–1405, Jun. 2007.
[128] P. A. Weiner, "Multidimensional convolutional codes," Ph.D. dissertation, Math. Dept., Notre Dame Univ., South Bend, IN, 1998.
[129] F. Winkler, *Polynomial Algorithms in Computer Algebra*. New York: Springer-Verlag, 1996.
[130] J. Wood, E. Rogers, and D. Owens, "A formal theory of matrix primeness," *Math. Control, Signals, Syst.*, vol. 11, pp. 40–78, 1998.
[131] J. Wood, E. Rogers, and D. Owens, "Minimum lag descriptions and minimal Gröbner bases," *Syst. Control Lett.*, vol. 34, no. 5, pp. 289–293, 1998.
[132] J. Wood, "Module and behaviors in nD systems theory," *Multidimensional Syst. Signal Process.*, vol. 11, pp. 11–48, 2000.
[133] J. Wu and L. Zhao, "Multi-valued model checking via Gröbner basis approach," in *Proc. IEEE/IFIP Theoretical Aspects of Software Engineering*, Jun. 6–8, 2007, pp. 35–44.
[134] L. Xu, J. Q. Ying, and O. Saito, "Feedback stabilization for a class of MIMO nD systems by Gröbner basis approach," in *Abstract 1st Int. Workshop Multidimensional Syst.*, Lagow, Poland, Jul. 1998, pp. 88–90.
[135] L. Xu, O. Saito, and K. Abe, "Output feedback stabilizability and stabilization algorithms for 2-D systems," *Multidimensional Syst. Signal Process.*, vol. 5, pp. 41–60, 1994.
[136] L. Xu, Z. Lin, and Y. Anazawa, "Minimal realization for a class of nD filters by Grobner basis approach," in *Proc. Asian Symp. Comput. Math.*, Seoul, Korea, Dec. 2005, pp. 185–188.
[137] L. Xu, "Applications of Gröbner bases in synthesis of multidimensional control systems," in *Proc. Workshop on GB in Contr. Theory Signal Process.*, Linz, Austria, May 2006.
[138] M. Yagi, T. Hisakado, and K. Okumura, "Algebraic representation of error bounds for describing function using Groebner base," in *Proc. ISCAS*, Kobe, Japan, May 2005, pp. 2831–2834.
[139] J. Q. Ying, L. Xu, and Z. Lin, "A computational method for determining strong stabilizability of n-D systems," *J. Symbol. Computat.*, vol. 27, pp. 479–499, 1999.
[140] D. C. Youla and G. Gnavi, "Notes on n-dimensional system theory," *IEEE Trans. Circuits Syst.*, vol. CAS-26, no. 2, pp. 105–111, Feb. 1979.
[141] D. C. Youla and P. F. Pickel, "The Quillen-Suslin theorem and the structure of n-dimensional elementary polynomial matrices," *IEEE Trans. Circuits Syst.*, vol. CAS-31, no. 4, pp. 513–518, Apr., 1984.
[142] G. Zacharias, "Generalized Gröbner basis in commutative polynomial rings," bachelor's thesis, Dept. Comput. Sci., MIT, Cambridge, 1978.
[143] E. Zerz, "Primeness of multivariate polynomial matrices," *Syst. Control Lett.*, vol. 29, pp. 139–145, 1996.
[144] E. Zerz, *Topics in Multidimensional Linear Systems Theory*. London, U.K.: Springer-Verlag London Ltd., 2000, vol. 256, Lecture Notes in Control and Information Sciences.
[145] E. Zerz, "Extension modules in behavioral linear systems theory," *Multidimensional Syst. Signal Process.*, vol. 12, pp. 309–327, 2001.
[146] E. Zerz and V. Lomadze, "A constructive solution to interconnection and decomposition problems with multidimensional behaviors," *SIAM J. Control Optim.*, vol. 40, no. 4, pp. 1072–1086, 2001.
[147] L. Zhang and A. Makur, "Two-dimensional antisymmetric linear phase filter bank construction using symmetric completion," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 54, no. 1, pp. 57–60, Jan. 2007.
[148] J. Zhou and M. N. Do, "Multidimensional multichannel FIR deconvolution using Gröbner bases," *IEEE Trans. Image Process.*, vol. 15, no. 10, pp. 2998–3007, Oct. 2006.



Zhiping Lin (SM'00) received the Ph.D. degree in information engineering from the University of Cambridge, Cambridge, U.K., in 1987.

He was with the University of Calgary, Calgary, AB, Canada, from 1987 to 1988, Shantou University, Guangdon, China, from 1988 to 1993, and DSO National Laboratories, Singapore, from 1993 to 1999. Since February 1999, he has been an Associate Professor with Nanyang Technological University (NTU), Singapore. He is also the Program Director of Bio-Signal Processing, Center for Signal Processing, NTU. He was an editorial board member of *Multidimensional Systems and Signal Processing* from 1993 to 2004 and has been a Co-Editor of the same journal since 2005. He has been an Associate Editor of *Circuits, Systems and Signal Processing* since 2000. His research interests include multidimensional systems and signal processing, array signal processing, and biomedical signal processing.

Dr. Lin is a Distinguished Lecturer of the IEEE Circuits and Systems Society for 2007–2008.

Li Xu (M'94) received the B.Eng. degree from Huazhong University of Science and Technology, Wuhan, China, in 1982, and the M.Eng. and Dr.Eng. degrees from Toyohashi University of Technology, Toyohashi, Japan, in 1990 and 1993, respectively.

From April 1993 to March 1998, he was an Assistant Professor with the Department of Knowledge-Based Information Engineering, Toyohashi University of Technology. From April 1998 to March 2000, he was a Lecturer with the Department of Information Management, Asahi University, Gifu, Japan. Since April 2000, he has been with the Faculty of Systems Science and Technology, Akita Prefectural University, Akita, Japan, where he is currently a Professor with the Department of Electronics and Information Systems. His research interests include multidimensional system theory, signal processing, and the applications of computer algebra to system theory. He has been an Associate Editor for *Multidimensional Systems and Signal Processing* (MSSP) since 2000 and was the Guest Co-Editor for the Special Issue of Applications of Grobner Bases in Multidimensional Systems and Signal Processing for MSSP published in 2001.

Nirmal Kumar Bose (LF'06) is the HRB-Systems Professor of Electrical Engineering with The Pennsylvania State University, University Park. He served as either a regular or visiting faculty for extended periods at several institutions, including the American University of Beirut, Lebanon, the University of Maryland, College Park, the University of California at Berkeley, Ruhr University (Germany), and Princeton University, Princeton, NJ. He was also invited for long-term visits to LAAS, Toulouse, France, the Centre for Artificial Intelligence and Robotics, Bangalore, India, Tokyo Institute of Technology (1999–2000), Tokyo, Japan, and Akita Prefectural University, Akita, Japan (2005) to conduct research and give seminars. He is the author of several textbooks in multidimensional systems theory, digital filters, and artificial neural networks. He is, since 1990, been the founding Editor-in-Chief of the *International Journal on Multidimensional Systems and Signal Processing* and has served on the editorial boards of several other journals.

Prof. Bose was the recipient of several honors and awards, including, more recently, the Invitational Fellowship from the Japan Society for the Promotion of Science in 1999, the Alexander von Humboldt Research Award from Germany in 2000, the Charles H. Fetter University Endowed Fellowship from 2001–2004 and the IEEE Circuits and Systems Education Award in May 2007.